Journal of Mathematical Behavior 59 (2020) 100795

Contents lists available at ScienceDirect

Journal of Mathematical Behavior

journal homepage: www.elsevier.com/locate/jmathb

# A quadratic growth learning trajectory☆

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A R T I C L E I N F O

*Keywords:*
Quadratic function
Learning trajectory
Quantitative reasoning
Representational fluency
Instructional supports
DNR

A B S T R A C T

This paper introduces a quadratic growth learning trajectory, a series of transitions in students' ways of thinking (WoT) and ways of understanding (WoU) quadratic growth in response to instructional supports emphasizing change in linked quantities. We studied middle grade (ages 12–13) students' conceptions during a small-scale teaching experiment aimed at fostering an understanding of quadratic growth as phenomenon of constantly-changing rate of change. We elaborate the duality, necessity, repeated reasoning framework, and methods of creating learning trajectories. We report five WoT: *Variation*, *Early Coordinated Change*, *Explicitly Quantified Coordinated Change*, *Dependency Relations of Change*, and *Correspondence*. We also articulate instructional supports that engendered transitions across these WoT: teacher moves, norms, and task design features. Our integration of instructional supports and transitions in students' WoT extend current research on quadratic function. A visual metaphor is leveraged to discuss the role of learning trajectories research in unifying research on teaching and learning.

## 1. Introduction and literature review

### *1.1. Students' learning of quadratic function*

Characterizing and supporting students' meaningful learning of function remains an important and challenging goal of algebra and algebraic thinking across school mathematics (e.g., Ayalon & Wilkie, 2019; Ellis, 2011b; National Governors Association & Council of Chief State School Officers, 2010; Stephens, Ellis, Blanton, & Brizuela, 2017). Non-linear functions in particular can be difficult for students to understand (Ellis & Grinstead, 2008; Lobato, Hohensee, Rhodehamel, & Diamond, 2012; Wilkie, 2019; Zaslavsky, 1997). This study addresses student learning of quadratic functions, an important topic in secondary school, in particular, when students begin to formally develop the algebraic tools to express and represent different functional relationships (Blanton et al., 2018; Stephens, Ellis et al., 2017). There are a host of studies that point to areas of difficulty students experience with understanding quadratic function (e.g., see Wilkie, 2019). In brief, students may struggle to interpret the coefficient of a quadratic and the role of parameters (Dreyfus & Halevi, 1991; Ellis & Grinstead, 2008; Zaslavsky, 1997), have difficulty moving among representations of quadratic functions (Kotsopoulos, 2007; Metcalf, 2007; Moschkovich, Schoenfeld, & Arcavi, 1993), may possess a compartmentalized, procedural view of quadratic functions (Parent, 2015), and often inappropriately generalize from linearity (Ellis & Grinstead, 2008; Schwarz & Hershkowitz, 1999; Zaslavsky, 1997). Despite these well documented difficulties, there are few studies that

☆ Support for this research was provided in part by the U.S. Department of Education IES Research Training Programs in the Education Sciences under grant no. R305B130007, and the National Science Foundation under Award REC 0952415. Any opinions, findings, and conclusions expressed in this material are those of the authors, not the funding agencies.

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https://doi.org/10.1016/j.jmathb.2020.100795
Received 13 October 2019; Received in revised form 3 June 2020; Accepted 8 June 2020

intentionally design instruction to address these challenges to students' learning of quadratic function.

One step toward addressing more robust supports for students' understanding of quadratic functions is to articulate instructional goals and design principles. Ellis' (2011a) study on students' generalization of quadratic growth, for instance, articulated the need to situate students' exploration within visualizable, manipulable contexts in order to identify constantly-changing rates of change of two co-varying quantities. Relatedly, Lobato et al.'s (2012) study elaborated five conceptual learning goals for quadratic function, as well as a set of instructional tasks that could elicit such student understandings. These learning goals included: (a) conceptualize change in dependent quantities, (b) conceive of changes in independent quantities and corresponding sets of changes in dependent quantities, (c) construct a sequence of ratios of change in dependent quantities to change in independent quantities, (d) construct rates of change as a new quantity, and (e) conceive the rate of rate of change of the dependent variable with respect to the independent variable as constant. Both of these studies stressed the importance of supporting students' understanding of quadratic growth within a quantitatively rich situation with a constantly-changing rate of change.

Lobato et al.'S (2012) study provided a viable framework for what students should learn with respect to quadratic function and recommended designing tasks that involve situations with linked quantities such as length-area and time-distance. Related studies have investigated instructional supports for quadratic functions, including principles for task design and the role of multiple representations. For example, in a study of students' intuitions of quadratic growth patterns, Wilkie (2019) explored secondary students' approaches to generalizing quadratic functions from figural growth patterns, and found that "[d]rawing on a visual context such as figural growing patterns might support students in learning conceptually about what actually makes a function quadratic in nature" (p. 16). Wilkie further suggested task design features such as sequencing tasks of the same type, asking students to generalize from visual patterns, and creating quadratic growth patterns by attending to bi-directional links across figures, equations, tables and graphs. In a related study, Selling (2016) argued that communicating with and reasoning about multiple representations supported students' learning of specific aspects of quadratic function such as first and second differences, and explicit and recursive rules. Finally, Rivera and Becker (2016) found that students were able to generalize quadratic growth patterns from a series of figural growth patterns with instructional support. Despite these advances, however, the nature of instructional supports remains underspecified, a gap this study aims to address.

### 1.2. Research aims

We build on the body of work that advances our understanding of both *what* students are to learn about quadratic functions, and *how* instructional supports may engender that learning. Our aim is to not only expand characterizations of students' learning of quadratic function (*what* students learn), but also to link these characterizations together with mechanisms of learning and instructional supports (*how* students learn). Our design and development inquiries were guided by the following research questions: *(a) How can middle-school students' learning of quadratic growth be characterized and supported? (b) How do goal-directed instructional supports engender that learning?*

To address these questions, we developed a quadratic growth learning trajectory as a representation of transitions in the mathematics of students and the instructional supports that engendered those transitions. We report five Ways of Thinking (WoT) students demonstrated about quadratic growth: *Variation*, *Early Coordinated Change*, *Explicitly Quantified Coordinated Change*, *Dependency Relations of Change*, and *Correspondence*. We also introduce three types of instructional supports that engendered transitions across these ways of thinking: teacher moves, norms, and task design features.

## 2. Theoretical framework

### 2.1. Defining a learning trajectory

The construct of a learning trajectory has been discussed in a variety of ways in the literature (Clements & Sarama, 2004; Ellis, Weber, & Lockwood, 2014; Fonger, Stephens et al., 2018; Lobato & Walters, 2017; Simon et al., 2010). Simon's (1995) original construct was for a hypothetical learning trajectory, which consisted of "the learning goal, the learning activities, and the thinking and learning in which students might engage" (p. 133). Some researchers emphasize the evolution of mental concepts as the key aspect of a learning trajectory. For instance, Wilson, Sztajn, and Edgington (2013) defined a learning trajectory as a research-based description of how students' thinking evolves over time from informal to more formal and complex mathematical ideas, Battista (2004) described increased levels of cognitive sophistication through which students progress until they reach formal concepts, and Hackenberg (2013) described a learning trajectory as a model of students' internal conceptions and an account of changes in their schemes and operations. Similarly, Confrey, Maloney, Nguyen, Mojica, and Myers (2009), Panorkou, Maloney, and Confrey (2013) depicted learning trajectories as emphasizing students' refinement of their own conceptual understanding.

Other researchers have emphasized the coordination of learning goals with instructional tasks and activities. Clements and Sarama (2004) described a learning trajectory as consisting of three parts: a mathematical goal, a model of cognition which they called developmental progressions, and instructional tasks providing experiences to support students' movement through the levels. This inclusion of tasks separates their definition from other progressions that document only sequences of students' thinking (e.g., Daro, Mosher, & Corcoran, 2011; National Assessment Governing Board [NAGB], 2008). Sarama (2018) has continued to emphasize the role of curricular tasks, describing a learning trajectory as a "device whose purpose is to support the research-grounded development of a curriculum or other unit of instruction" (p. 72).

Research on learning has largely developed separately from research on teaching (Myers, Sztajn, Wilson, & Edgington, 2015), but

Sarama (2018) cautioned that one cannot discount the role of instruction. Similarly, Confrey et al. (2009) have explicitly acknowledged that conceptual growth, as depicted in a learning trajectory, is influenced by instruction. Simon et al. (2010) have called attention to a paradox in studying learning—that to study learning, instruction must promote the learning one intends to study. Steffe (2004) also directly addressed the role of instruction by acknowledging the importance of accounting for changes in students' concepts and operations "as a result of children's interactive mathematical activity in the situations of learning, and an account of the mathematical interactions that were involved in the changes" (p. 131). The few studies of learning trajectories that directly address instructional actions typically attend to a sequence of learning goals and instructional activities (Fonger, Davis, & Rohwer, 2018, 2018b), instructional practices and activity structures (Fonger, 2018; Stephens, Fonger et al., 2017), the nature of shifts in student conceptions during instructional interventions (Ellis, Ozgur, Kulow, Dogan, & Amidon, 2016), and students' activity and engagement with mathematical tasks (e.g., Simon et al., 2010). Following these approaches, we define a learning trajectory to be an empirically-based model of students' understandings, along with an account of changes in understanding in relation to students' interaction with instructional supports including mathematical tasks, tools and representations, and teacher moves (Ellis et al., 2016).

### 2.2. DNR-based instruction

We draw on DNR-based instruction (Harel, 2008a, 2008b) in order to inform our instructional design principles for the development, enactment, and analysis of our learning trajectory. DNR stands for the three instructional principles *duality*, *necessity*, and *repeated reasoning*. The duality principle addresses two forms of knowledge, *Ways of Understanding* (WoU) and *Ways of Thinking* (WoT). WoU are students' concepts of specific subject matter, including particular definitions, theorems, proofs, problems, and their solutions (Harel, 2008a). For instance, one WoU about quadratic functions is that quadratic growth is a representation of a relationship between two co-varying quantities in which one quantity varies at a constantly-changing rate of change relative to the other. WoT are broader conceptual tools, such as empirical reasoning, deductive reasoning, heuristics, and beliefs about mathematics (Harel, 2013). For instance, one WoT relevant to students' learning of quadratic functions is a treatment of functional relationships as number patterns, devoid of quantitative referents. Alternatively, one could have a WoT that functions can be investigated as a phenomenon of co-varying quantities. The duality principle states that students develop WoT through the production of WoU, and, conversely, the WoU they produce are afforded and constrained by the WoT they possess (Harel, 2008a). The central objective of the initial content of a learning trajectory, then, must be formulated in terms of both WoU and WoT.

The necessity principle addresses the notion of intellectual need, stating that in order for students to learn the mathematics we intend to teach them, they must experience a need for it (Harel, 2008b). Intellectual need can be engendered through situations that students experience as problematic and that necessitate the creation of new knowledge in order to be resolved. This requires developing problem tasks that students can relate to and become invested in and that necessitate the development of new WoU as a consequence of intellectual engagement with the task, rather than through students' social needs, such as the need to please their teacher, or to achieve a high grade.

Finally, the repeated reasoning principle addresses the importance of ensuring that students internalize, retain, and organize their knowledge (Harel, 2008a). Repeated reasoning is a mechanism for reinforcing desirable WoU and WoT. Repeated reasoning should not, however, be confused with the drill and practice of routine problems. Rather, it is an instructional principle that relies on providing students with sequences of problems that continually call for thinking through puzzling situations and solutions; the problems respond to students' intellectual needs.

### 2.3. Quantitative reasoning and rates of change

Two common approaches that typically undergird function instruction are the correspondence approach and the coordination or covariation approach. The correspondence approach, which is overwhelmingly prevalent in secondary mathematics curricula and standards in the United States (National Governors Association & Council of Chief State School Officers, 2010; Thompson & Carlson, 2017), emphasizes functional relationships as mappings. A function $y = f(x)$ is defined as a relation between members of two sets, with each value of $x$ mapped to a unique value of $y$ (Smith, 2003). This approach emphasizes the development of closed-form, explicit rules that can be used to analyze and predict function behaviors.

In contrast, Confrey and Smith (1994), Saldanha and Thompson (1998), Smith (2003), Smith and Confrey (1994) and Thompson and Thompson (1992), Thompson (1994), Thompson and Carlson (2017) introduced what they called covariation approaches, although these approaches differ somewhat from one another. Confrey and Smith's (1994) covariation approach emphasizes attention coordinated changes in $x$- and $y$-values, in which one can move operationally from $y_m$ to $y_{m+1}$ coordinated with movement from $x_m$ to $x_{m+1}$. This approach supports engagement with tables and graphs that can be interpreted by students as presenting successive states of variation, supporting the development of an understanding that quantities have sequences of values. We call this way of thinking the *coordination approach*, and note that students can attend to coordinated changes in co-varying quantities even if their images of quantitative situations are static, rather than dynamic.

In a different approach, Saldanha and Thompson (1998) addressed the imagistic foundations that support students' abilities to reason about quantities that vary, either independently or simultaneously. This form of reasoning involves mentally holding a sustained image of two quantities' values simultaneously while attending to how they change in relation to one another (Castillo-Garsow, 2013). From this perspective, covariational thinking entails the coupling of two quantities in order to form a multiplicative object; once this object is formed, a student can attend to either quantity's value with the explicit understanding that at every instance, the other quantity also has a coordinated value. This approach addresses the importance of developing dynamic, rather than

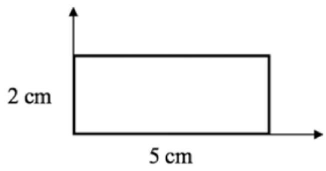



**Fig. 1.** A diagram of the proportionally growing rectangle context.

static images of quantitative situations, and we call it the covariation approach to distinguish it from the coordination approach described above.

The covariation and coordinated change approaches leverage the notion of a quantity, which Thompson (1994) defined as a person's scheme composed of an object, a quality of the object, an appropriate unit or dimension, and a process for assigning a numerical value to the quality. Thus, a quantity is a conceptual entity, rather than a characteristic that exists in the object itself. Attributes such as length, area, distance, and speed can be conceived as quantities. Quantitative reasoning is the process of reasoning with quantities, their relationships, and associated mathematical operations.

## 3. Methods

This study is situated in the paradigm of design-based research, in which our goal is to simultaneously engender innovative forms of learning and to study the resulting learning in the context in which it was supported (Cobb & Gravemeijer, 2008; Cobb, diSessa, Lehrer, & Schauble, 2003; Gravemeijer & Cobb, 2006). A design experiment has three phases: *design*, *experiment*, and *analyze*.

### *3.1. Design*

In the design phase we created a hypothetical learning trajectory (Simon, 1995) of goals, potential instructional supports, and hypotheses of students' learning processes informed by the literature on students' learning of quadratic function, DNR-based instruction design principles, and information learned from pre-interviews with each student. Prior research has documented the difficulties in supporting students' meaningful understandings of quadratic function from approaches that emphasize patterning, and representational forms of quadratic functions (see Section 1.1). In contrast, we chose to leverage research and theory on quantitative reasoning and rates of change (see Section 2.3) to ground the design of a hypothetical learning trajectory.

Our goal was to engender students' understanding of quadratic growth as a phenomenon of constantly-changing rate of change among linked quantities (Ellis, 2011a; Saldanha & Thompson, 1998). In order to support this goal, we designed a context with quantities that students could visualize, manipulate, and explore by comparing the lengths, heights, and areas of proportionally-growing rectangles (Ellis, 2011a). These rectangles grew in both length and height but maintained the ratio of length to height ($a$); thus, the relationship between the height, $h$, and the area, $A$, can be expressed as $A = ah^2$ (Fig. 1). Using dynamic geometry software, students could manipulate and measure a rectangle's length, height and area to discern the relationship between these changing quantities.

Within this instructional context, we sought to engender three WoU: (a) the rate of change of a rectangle's area grows at a constantly-changing rate compared to changes in height; (b) given a height, $h$, the rectangle's area can be determined by $A = ah^2$ where $a$ is the ratio of length to height; and (c) the constantly-changing rate of change of the area, $A$, is dependent on the change in height ($\Delta h$) such that for the ratio of length to height ($a$), the constantly-changing rate of change is $2a(\Delta h)^2$.[1] We designed a series of tasks that we conjectured would engage students in quantitative reasoning. For example, we asked students to draw several iterations of the proportionally growing rectangle and to create tables relating changes in height and area. Other tasks prompted for generalization of relationships (see Fig. 2).

### *3.2. Experiment*

#### *3.2.1. Setting and participants*

The study was situated at a public middle school located in a midsized city in the United States. The participants were 6 eighth-grade students (3 girls and 3 boys) who were enrolled in pre-algebra (3 students), algebra (2 students), and geometry (1 student). The students' teachers identified them as either high, medium, or low based on their assigned mathematics class, as well as on their mathematics grades, attendance, and participation in class. Two students were identified as high, 2 students were identified as medium, and 2 students were identified as low. One student was Indian American, 2 students were Asian American (one of them was an English language learner), and 3 students were Caucasian. Gender-preserving pseudonyms were used for all participants. None of those students had experience with quadratic functions, but all had studied linear functions in their courses.

[1] For an elaboration of the mathematics of this generalized case, see Ellis (2011a).

| | |
|---|---|
| Make a table for the height, length, and area of the 1x2 rectangle as it grows.<br><br>Draw each rectangle. How many squares are being added on each time? | Here is a table for the height versus the **AREA** of a rectangle that is growing in proportion. Find the missing quantities.<br>Height — Area<br>3 — 6.75<br>4 — 12<br>5 — 18.75<br>6 — 27<br>7 — 36.75<br>8 — 48<br>50 — ?<br>h — ? |

**Fig. 2.** Two sample tasks.

| Day | Mathematical Topics | Day | Mathematical Topics |
|---|---|---|---|
| 1 | Measurement and area | 9 | Justifying the second differences as $2a$ |
| 2 | Comparing perimeter and area | 10 | Identifying second differences for tables with different $\Delta h$ values |
| 3 | Identifying first and second differences in tables | 11 | Connecting equations, tables, and graphs |
| 4 | Connecting first and second differences to area | 12 | Graphing parabolas |
| 5 | Identifying height : length ratios and creating $y = ax^2$ equations | 13 | Graphing first and second differences |
| 6 | Creating generalizations about second differences | 14 | Creating $y = ax^2 + c$ equations and graphs |
| 7 | Justifying generalizations about second differences | 15 | Summarizing generalizations |
| 8 | Creating $y = ax^2$ equations from tables and identifying the second differences as $2a$ | | |

**Fig. 3.** Overview of the TE sessions (reproduced from Ellis, 2011a, p. 313).

*3.2.2. Teaching experiment*

We conducted a 15-session videoed teaching experiment [TE] (Cobb & Steffe, 1983), with each session lasting 1 h. The TE setting allowed for the creation and testing of hypotheses in real time while engaging in teaching actions. This meant that the mathematical topics for the entire set of sessions were not predetermined but instead were created and revised on a daily basis in response to hypothesized models about the students' mathematics. Fig. 3 provides a brief overview of the topics addressed in the TE. Task design was also an ongoing part of the experiment. For example, as we learned new information about student learning processes, we designed new tasks to support the students' intellectual need to elaborate and justify relationships that held for *all* proportionally growing rectangles.

The second author was the teacher-researcher (TR), and two project members observed each teaching session. The TR fostered a learning environment that encouraged students to make and test conjectures, to make predictions and generalizations, and to explicitly attend to quantities and their relationships. During TE activities, the students worked individually, in pairs, and in small groups, before then discussing their ideas with the entire group. The project members operated two video cameras during the teaching sessions to capture both the whole-group discussions and the small group interactions. The project team met after each session to debrief, discuss what had occurred during the session, and design new tasks or revisions to planned tasks.

### *3.3. Analyze*

Our goal in creating a learning trajectory was to establish rich descriptions of the nature of learning in the context of instructional supports (cf. Cobb & Gravemeijer, 2008). To attain this goal, we conducted several rounds of data analysis with differing foci. Our primary focus was to articulate change in the mathematics of students[2] ; our secondary focus was to contextualize and explain these changes with respect to instructional supports. All sessions of the TE were transcribed; then, all transcripts were enhanced to include verbal utterances, images of the drawings on the board, student's written predictions prior to discussion, and descriptions of the students' gestures. To analyze the data, we applied the constant comparative method (Strauss & Corbin, 1990) and axial coding (Strauss, 1987) to develop and discern relationships among codes.

In the first round of coding, all three authors independently coded the first eight sessions and then met to discuss and reconcile

[2] We distinguish between first-order knowledge (one's own knowledge) and second-order knowledge (a model of another's knowledge) (cf. Steffe, von Glasersfeld, Richards, & Cobb, 1983). In this paper we aim to build second-order models of students' first order knowledge. Said otherwise, *students' mathematics* entails "a students' first-order mathematical knowledge" while the *mathematics of students* entails "explanatory models of students' mathematics" (Steffe, n.d., p. 7). Thus, in our focus on building models of students' mathematics, we are characterizing the mathematics of students.

their coding decisions. During this phase, we established an initial framework of emergent categories and subcategories of relevant student concepts. In the second phase, we applied our coding framework to sessions 9–15, again meeting to reconcile our coding after each session. These reconciliations resulted in modifying code descriptions, adding new codes, and collapsing existing codes. After these two rounds, we reached a stable coding scheme and then re-coded all sessions with the final scheme, which included specific WoU and broader WoT. As a note, Harel used the terms WoU and WoT to distinguish specific mathematical ideas from broader heuristics and beliefs. In contrast, we have adapted these terms to distinguish clusters of concepts as WoT, and then the specific concepts within each cluster as WoU. In order to distinguish how we have adapted Harel's term "WoT" to identify these clusters, we will hereafter refer to them by number: *WoT1*, *WoT2*, etc. These numbered WoT are distinguished from the more general WoT that are not part of the learning trajectory, but are instead heuristics and beliefs, such as attending to patterns.

In the final phase of coding, we identified transitions in students' WoT and how those shifts occurred in response to instructional supports. We identified an *instructional support* by analyzing the data corpus for evidence of task features, teacher moves, and other features that appeared to engender students' WoT and WoU. While the instructional supports were emergent, they were also informed by our knowledge of the existing literature on teacher moves (e.g., Bishop, Hardison, & Przybyla-Kuchek, 2016; Franke et al., 2009; Peterson et al., 2017), *norms* (Cobb & Yackel, 1996; Stephan, 2014), and *tasks* (Bieda & Nathan, 2009; McCallum, 2019; Stein & Smith, 1998).

## 4. Results: the quadratic growth learning trajectory

In this section we introduce models of students' WoT and WoU about quadratic growth, the transitions that occurred from one WoT to the next, and the instructional supports that engendered those transitions. We identified five WoT: *WoT1 Variation*, *WoT2 Early Coordinated Change*, *WoT3 Explicitly Quantified Coordinated Change*, *WoT4 Dependency Relations of Change*, and *WoT5 Correspondence*. Each of these five WoT includes a number of specific WoU. We also found three types of *instructional supports*, which we categorize into (a) teacher moves, (b) norms, and (c) task design features.

A *teacher move* is what the TR does in response to students' mathematical thinking, or to elicit new instances of mathematical thinking when engaging an individual student, a small group, or the whole class (cf. Peterson et al., 2017). We found these teacher moves to fall under three broad design heuristics: quantitative reasoning, representational fluency, and generalization. *Norms* are the expectations that the teacher and students have for each other that are present during mathematical discussion or student engagement with mathematical tasks (cf. Cobb & Yackel, 1996). We found the norms to cluster around two design heuristics—quantitative reasoning and representational fluency—as well as social norms for mathematical discourse in classroom interaction. A *task* is statement of a mathematical problem or set of problems that focuses students' attention on a particular mathematical idea or provides an opportunity for students to engage in a particular way of thinking (cf. Stein & Smith, 1998). We found several *task design features* and characterized these as falling into three types: doing mathematics, far prediction and generalization, and repeated reasoning. Sample tasks are given in the methods section and the forthcoming results subsections.

The learning trajectory for quadratic growth is introduced in Table 1. The learning trajectory includes the mathematics learning goal, the mathematics of students—both *WoT1* through *WoT5* and their subsequent *WoU*—and the instructional supports that engendered transitions in the mathematics of students. We caution the reader *not* to interpret the learning trajectory as a set of understandings that were predetermined, nor as a list of understandings and instructional supports that occurred in a linear order. Instead, this learning trajectory constitutes a model characterizing how the students learned to organize their WoT and related WoU in an intentionally designed, supportive instructional context guided by a conceptual learning goal. Fig. 4 introduces a visualization of the set of instructional supports unified with the WoT and WoU. This depicts the quadratic growth learning trajectory as a dynamic model of transitions.

In the following sections we elaborate the four major transitions the students experienced from one WoT to the next. Each section addresses the relevant WoT with select examples of WoU and the instructional supports that engendered the students' transitions.

### *4.1. Transition I: from variation to early coordinated change*

The first transition in the learning trajectory is from *WoT1 Variation,* to *WoT2 Early Coordinated Change*. The students' initial attention to variation was such that they did not coordinate change across quantities. Instead, they only attended to variation in one quantity at a time, or, they attended to two types of variation but only as a sequence of disconnected changes.

#### *4.1.1. WoT1 variation*

Initial tasks for the quadratic growth situation incremented a proportionally growing rectangle by 1 cm in height, and we then began to introduce tasks in which the rectangles' height or length values grew by greater than 1 cm. The students created their own data tables to keep track of the growth in height, length, and area. For example, given a growing square, Jim created a table relating the square's length and width to its area (Fig. 5). When the TR asked Jim to discuss his table, Jim focused solely on describing the area's second differences of 18 $cm^2$ as "going up by 18 s" without attending to how the square's height (or length) grew. We call this *WoU1.1 Single-Quantity Variation* because the students noticed and described changes in one quantity's magnitude without attending to the other quantity or changes in its magnitude. We note that it is likely that the students did not even implicitly attend to changes in the other quantity's magnitude because they did not experience an intellectual need to do so, given that their tables were well-ordered.

Another observed WoU within *WoT1* is *WoU1.2 Uncoordinated Variation*, in which students began attending to variation across

**Table 1**
A Quadratic Growth Learning Trajectory.

| Conceptual Learning Goal | | | |
|---|---|---|---|
| Support students' understanding of quadratic growth as a relationship between quantities such that the dependent quantity $y$ has a constantly-changing rate of change with respect to the independent quantity $x$. In symbols, for a quadratic function $f: x \rightarrow y$, if $\Delta\Delta y = d$, then $y = d/(2(\Delta x)^2) \cdot x^2$ | | | |
| Students' Ways of Thinking (WoT), Ways of Understanding (WoU) | Instructional Supports | | |
| | ***Teacher Moves*** | ***Norms*** | ***Task Design Features*** |
| **Initial**<br>*WoT1 Variation*<br>• WoU1.1 Single-Quantity Variation<br>• WoU1.2 Uncoordinated Variation | | | |
| **Transition I**<br>*WoT2 Early Coordinated Change*<br>• WoU2.1a,b Implicit Coordinated Change<br>• WoU2.2 Qualitative Coordinated Change | ***Quantitative Reasoning:***<br>• Ask students to explicitly attend to how quantities change together as expressed in diagrams, words, and tables;<br>• Explicitly draw attention to and identify quantities; and<br>• Press for quantitatively-based justifications. | ***Quantitative Reasoning:***<br>• Be explicit about the amount of change for each of the relevant quantities;<br>• Attend to linked quantities as relationships of explicit coordinated change; and<br>• Make and test predictions about generalized relationships among linked quantities. | ***Repeated Reasoning:***<br>• Investigate a quadratic growth situation where change in independent quantity ($\Delta x$) is greater than 1;<br>• Investigate a quadratic growth situation where change in independent quantity ($\Delta x$) is less than 1;<br>• Investigate a collection of tables for quadratic growth where the change in the independent quantity ($\Delta x$) is greater than 1 and varied across the set; and<br>• Sequence tasks of the same type to allow students to test conjectures about generalized relationships. |
| **Transition II**<br>*WoT3 Explicitly Quantified Coordinated Change*<br>• WoU3.1a,b Single-Unit Explicit Coordination<br>• WoU3.2a,b Multiple-Unit Explicit Coordination<br>• WoU3.3a,b Partial-Unit Explicit Coordination | ***Representational Fluency:***<br>• Ask students to create their own tables of two or more quantities changing together;<br>• Prompt students to connect different representations;<br>• Encourage students to create drawings or diagrams and to name quantities; and<br>• Encourage students to create graphs. | ***Representational Fluency:***<br>• Create tables and drawings to express relationships in quadratic growth contexts. | ***Far Prediction and Generalization:***<br>• Predict the changing rate of change in a quadratic growth situation from a table of quantities;<br>• Elicit conjectures about generalized relationships among linked quantities for a quadratic growth context;<br>• Make far predictions of independent and dependent quantities for a given quadratic growth pattern; and<br>• Elicit generalizations of relationships between independent and dependent quantities using variables. |
| **Transition III**<br>*WoT4 Dependency Relations of Change*<br>• WoU4.1 Recognition that Change in One Quantity Determines Change in the Other<br>• WoU4.2 Identification of How Change in One Quantity Determines Change in the Other<br>• WoU4.3 Translations of Dependency Relations of Change to Correspondence Rules | ***Generalization:***<br>• Prompt students to make conjectures and test their conjectures in a new task context; and<br>• Ask students to generalize a mathematical relationship they identified. | ***Mathematical Discourse:***<br>• Multiple student responses are shared for a single question or task; and<br>• Students respond to and build on each other's ideas. | ***Doing Mathematics:***<br>• Encourage thinking through puzzling situations and solutions (e.g., given an un-specified quadratic growth pattern); and<br>• Quadratic growth task allows for multiple different ways of interpreting and explaining one's reasoning. |
| **Transition IV**<br>*WoT5 Correspondence*<br>• WoU5.1a,b Correspondence Relations Between Independent and Dependent Quantities | | | |

multiple quantities as isolated patterns. This was particularly prominent in far prediction tasks for tables in which the height was incremented by more than 1 cm. For example, given the table in Fig. 6, Ally conceived of the difference in length as "going up by eight" and also noted that "It's going up by 2 s (in height)," yet did not attend to how the two quantities changed together.

In Table 2 we summarize the two component WoU for *WoT1 Variation*. Notice in this table that each instructional support is identified for each WoU, one by one. All of the students entered the TE attending to changes in the values of one or more quantities; we saw this as evidence of a general WoT that valued pattern-seeking. The students were adept at finding and identifying many different patterns, particularly patterns within one quantity. This WoT supported the development of the students' initial *WoT1* of attending to single-quantity or disconnected variation. Thus, in task situations such as that given in Fig. 6, coordinated variation did

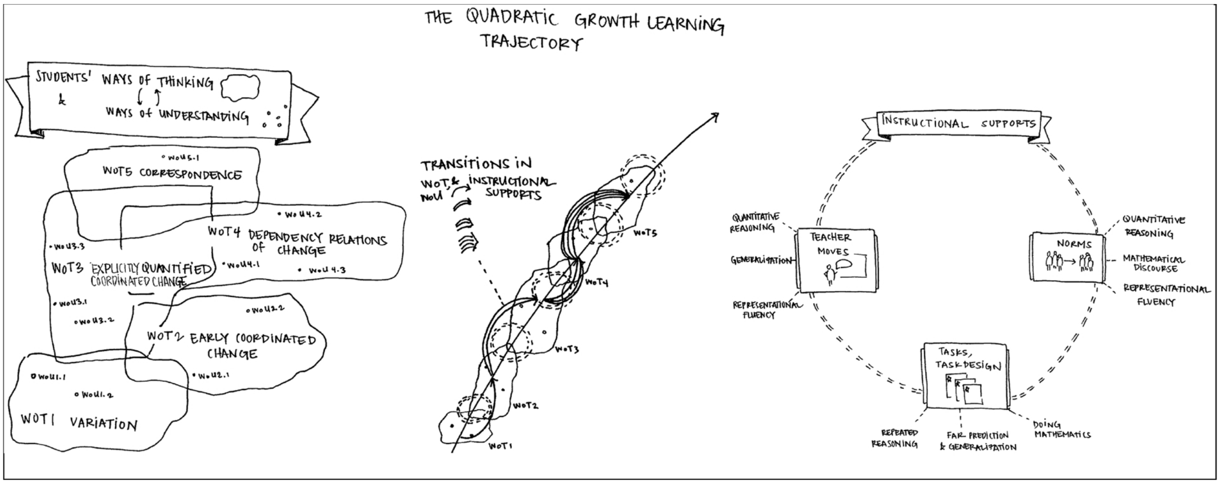



**Fig. 4.** A visualization of the quadratic growth learning trajectory as a dynamic model of transitions in students' WoT and WoU together with an integrated set of instructional supports.

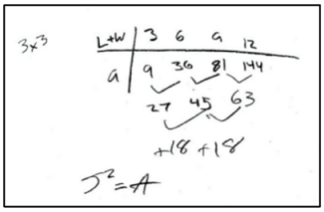


**Fig. 5.** Jim attended to single-quantity variation.

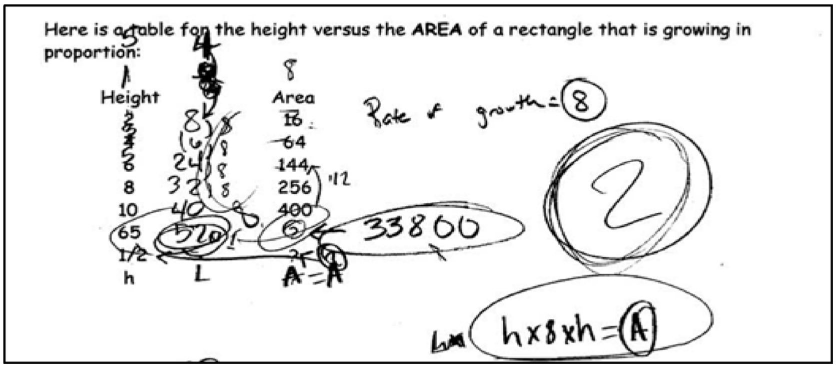



**Fig. 6.** Ally's uncoordinated variation of quantities.

**Table 2**
WoT1 Variation, Related WoU, and Instructional Supports.

| WoU | Definition | Data Example | Instructional Support |
|---|---|---|---|
| *WoT1 Variation* (Conceiving of single or multi-quantity variation without coordination) | | | |
| *WoU1.1 Single-Quantity Variation* | Student attends to a change in one quantity without coordinating this change with any other quantity change. | Jim: "This one is going up by 18 s." | *Task Design Feature*: Ask students to investigate a quadratic growth situation given an initial independent quantity ($x$) greater than 1. |
| *WoU1.2 Uncoordinated Variation* | Student attends to a change in more than one quantity without coordinating simultaneous change; variation is treated as isolated patterns or sequences. | Ally: "I figured out it was going up by eight (in length)" and "It's going up by 2 s (in height)." | *Teacher Move:* Ask students to explicitly attend to how quantities change together as expressed in diagrams, words, and tables. |

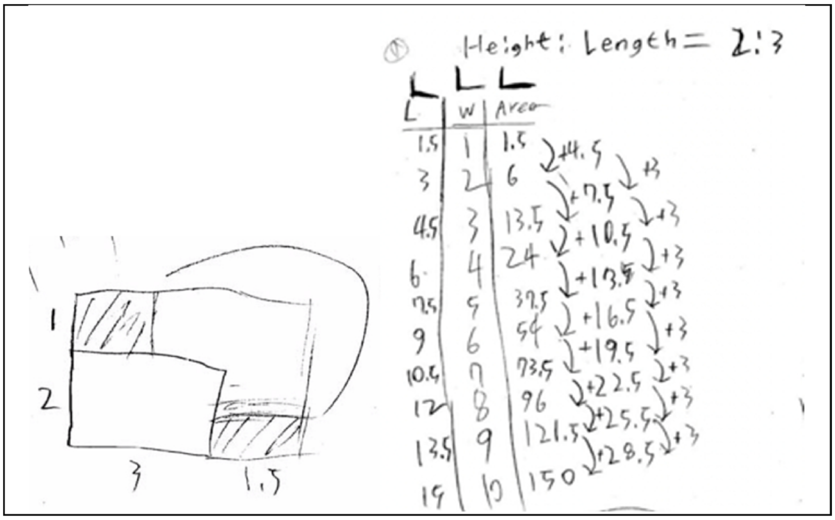



**Fig. 7.** Daeshim's drawing and table of the proportionally growing 2 cm by 3 cm rectangle.

not occur spontaneously for students. We found that several instructional supports were necessary for students to develop attention to coordinating changes in the values of two or more quantities, as we describe in the next section.

*4.1.2. WoT2 early coordinated change*

When students began to attend to coordinated change in the values of two or more co-varying quantities, we found that their initial conceptions of these changes were either qualitative (unquantified) or implicit for at least one of the quantities. We call this *WoT2 Early Coordinated Change*. We identified two main WoU for *WoT2*: *Implicit Coordinated Change (WoU2.1a,* and *WoU2.1b)* and *Qualitative Coordinated Change* (*WoU2.2*). Additionally, we found five instructional supports for developing *WoT2*: *teacher moves* of (1) asking students to explicitly attend to how quantities change together as expressed in diagrams, words, and tables, (2) press for quantitatively-based justifications, (3) prompts to connect different representations; a series of *tasks* with task design feature that afforded (4) investigation of a quadratic growth situation where change in independent quantity ($\Delta x$) is greater than 1, and (5) *tasks* that encouraged thinking through puzzling situations and solutions (e.g., with an un-specified growth pattern). As we exemplify below, we found certain instructional supports to engender particular WoU. Moreover, some instructional supports engendered more than one WoU within *WoT2*.

*4.1.2.1. WoU2.1 implicit coordinated change*. During the TE, the TR encouraged the students to describe how the values of the growing rectangle's quantities height, length, and area changed, as well as to explicitly connect how these quantities were interpreted in tables, figures, and words. For example, students investigated a proportionally growing rectangle that maintained a 2:3 ratio of height to length on a Geometer's Sketchpad © file that could be dragged to show dynamic growth. Daeshim's drawing and table are given in Fig. 7. Prompted to explain the table, Jim described "then [the] branch off of those is that they all go up by 3" (see the + 3 on the right side of the table in Fig. 7). The TR pressed students to explain this finding:

TR: Can you anticipate what I'm going to ask you now?

Jim: Where does the 3 come from?

TR: Where does the 3 come from? What does the 3 have to do with the picture?

Anna: Uh, this is just a guess, but uh, when it goes up by 3 s, like it did with the 2 s, like every time it grows it adds 3.

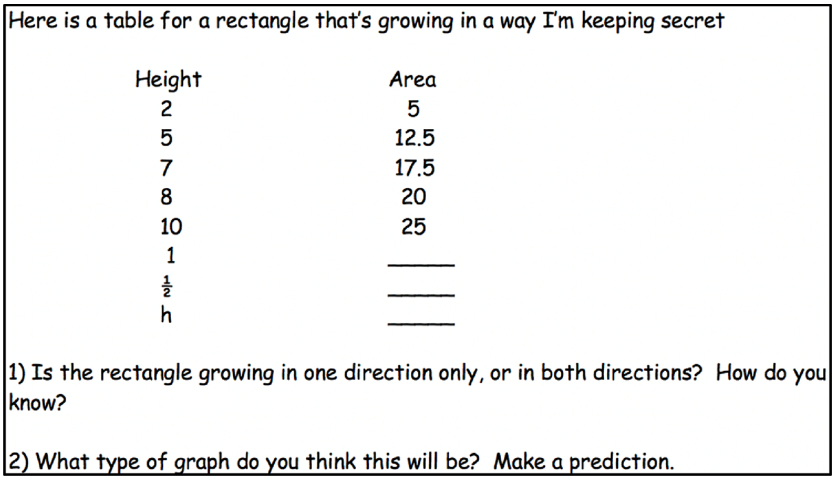
In this excerpt, the TR pressed the students to give a quantitatively based explanation for their finding, and Anna coordinated the constantly-changing difference in change in area (3 more squares) implicitly with a change in height, stating, "*every time* it grows, it adds 3." We take this as evidence of *WoU2.1b Implicit Coordination of Second Differences with Change in Another Quantity*. Note that *WoU2.1 Implicit Coordinated Change* includes two sub-levels; *WoU2.1a* focuses on implicit coordinated change in two quantities, while *WoU2.1b* focuses on coordinated change of second differences with change in another quantity (see Table 3).

*4.1.2.2. WoU2.2 qualitative coordinated change*. The second WoU for *WoT2* is *WoU2.2 Qualitative Coordinated Change*. An example of this WoU occurred when the students examined a "Mystery Table" (see Fig. 8), in which the area function was unknown. After the students worked on the task, the TR invited students to imagine how the height and length of the rectangle was growing based on their reading of the table.

TR: Is the rectangle growing in one direction only or do you think it's growing in both directions?

**Table 3**
WoT2 Early Coordinated Change, Related WoU, and Instructional Supports.

| *WoT2 Early Coordinated Change* (Conceiving of multi-quantity variation without explicit quantification of both quantities) | | | |
|---|---|---|---|
| WoU | Definition | Data Example | Instructional Support |
| *WoU2.1a Implicit Coordinated Change in Two or More Quantities.*<br>*WoU2.1b Implicit Coordination of Second Differences with Change in Another Quantity.* | Students attend to growth in both quantities together, but the magnitude of change remains implicit for one or both quantities.<br>Students coordinate the difference in the rate of growth of the area, but the magnitude of the change remains implicit for one or both quantities. | Jim: "How many new squares it's gaining every time it grows."<br>Anna: "like every time it grows it adds 3 [more squares]" | *Teacher Moves:*<br>● Ask students to explicitly attend to how quantities change together as expressed in diagrams, words, and tables; and<br>● Press for quantitatively-based justifications.<br>*Task Design Feature:*<br>● Investigate a quadratic growth situation where change in independent quantity ($\Delta x$) is greater than 1. |
| *WoU2.2 Qualitative Coordinated Change* | Student links the change in two or more quantities, understanding that they change together, without quantifying the change. | Daeshim: "Well, it's the length times height. If length were growing, area will be bigger". | *Teacher Moves:*<br>● Ask students to explicitly attend to how quantities change together as expressed in diagrams, words, and tables; and<br>● Prompt students to connect different representations.<br>*Task Design Feature:*<br>● Encourage thinking through puzzling situations and solutions (e.g., given an un-specified growth pattern). |


Here is a table for a rectangle that's growing in a way I'm keeping secret

| Height | Area |
|---|---|
| 2 | 5 |
| 5 | 12.5 |
| 7 | 17.5 |
| 8 | 20 |
| 10 | 25 |
| 1 | ____ |
| $\frac{1}{2}$ | ____ |
| h | ____ |

1) Is the rectangle growing in one direction only, or in both directions? How do you know?

2) What type of graph do you think this will be? Make a prediction.

**Fig. 8.** A Mystery Table Task.

Bianca: Both.

TR: Both? How come?

Jim: Because the height and length are changing in numbers.

Jim said that both the height and length were changing, but he did not quantify the change. This example illustrates that as the students began to attend to changes in two or more quantities simultaneously, they would often describe these changes without quantifying the nature of the change. In another example, Daeshim said, "Well, it's the length times height. If length were growing, area will be bigger." The instructional supports that encouraged attention to simultaneous (albeit qualitative) change included a teacher move of eliciting students' descriptions for how the rectangle grew, as well as the use of tasks that relied on "Mystery Tables". These tables provided students with height and area values and challenged them to determine the nature of the rectangle's growth, necessitating attention to how both quantities grew together. Another teacher move that encouraged the development of *WoT2* was a prompt to connect different representations. For instance, the TR encouraged the students to use the data provided in tables to draw rectangles with labelled quantities of height and length.

As summarized in Table 3, students' transition to *WoT2 Early Coordinated Change* was characterized by either *Implicit Coordinated Change* (*WoU2.1a* or *WoU2.1b*) or a *Qualitative Coordinated Change* (*WoU2.2*). We found the same collection of instructional supports to engender both *WoU2.1a* and *WoU2.1b*, hence the grouping across these rows in Table 3. Instructional supports included teacher moves such as prompting students to identify and articulate changes in quantities across figures, tables, graphs, as well as tasks that: (a) varied the initial value of the height, and (b) encouraged attention to two quantities changing together (e.g., the Mystery Table).

### 4.2. Transition II: establishing explicitly quantified coordinated change

The second major transition in the learning trajectory occurred when the students developed the *WoT3 Explicitly Quantified Coordinated Change*. By explicit quantified coordination, we mean that the students identified the amounts of change in each of the quantities as they coordinated changes in both quantities together. This WoT includes three main WoU: (a) *Single-Unit Explicit Coordination* (*WoU3.1a* and *WoU3.1b*), (b) *Multiple-Unit Explicit Coordination* (*WoU3.2a* and *WoU3.2b*), and (c) *Partial-Unit Explicit Coordination* (*WoU3.3a* and *WoU3.3b*). We found seven instructional supports that fostered these WoU. They included the *teacher moves* of: (1) explicitly drawing attention to and identifying quantities, (2) prompting students to create their own tables of two or more quantities that changed together, (3) pressing for quantitatively-based justifications, and (4) encouraging students to create drawings, diagrams, or graphs and name quantities. The instructional supports also included *tasks* requiring students to: (5) predict the constantly-changing rate of change, and (6) investigate tables of linked quantities with increments less than 1 for the independent variable. A *norm* was also established: (7) be explicit about the magnitude of change for each of the relevant quantities. We elaborate the specific links between this set of instructional supports and the particular WoU these instructional supports engendered in the following subsections.

#### 4.2.1. WoU3.1 single-unit explicit coordination

As discussed above, when the students initially began to coordinate changes in co-varying quantities, they did so without explicitly attending to the amount of change. In an attempt to encourage the students to explicitly attend to both the relevant quantities and the nature of their coordinated changes, the TR began to model this type of attention focusing through the teacher moves of drawing figures and naming quantities. She also asked the students to create their own tables to compare the rectangle's length to its area. The manner in which the table was organized (what values to begin with and what the amount of increase should be from one entry to the next) was left open to the students. In creating their own tables and making decisions about both initial values and amounts of increase, the students gained facility with explicitly attending to coordinated changes.

Recall Daeshim's table and figure for a proportionally growing 2 cm by 3 cm rectangle (Fig. 7). As the students created drawings and tables, Bianca said, "I found that the length is 1.5 times the height." The TR asked her to explain on the board with a picture. Bianca drew a growing rectangle similar to that shown in Daeshim's drawing on the left hand side of Fig. 7. She then explained, "If you increase it by 1 (adds on to her rectangle), then you've got, now it's 3 by one point, er, 4.5. So 3 times 1.5 equals 4.5. It just keeps going like that." As Bianca grew the rectangle in her drawing, she was forced to coordinate growth in height with growth in length, realizing that for every 1-cm growth in height, there was a corresponding 1.5-cm growth in length: "For every 1 you add it up, then you add 1.5 across." This is evidence of *WoU3.1a Single-Unit Explicit Coordinated Change in Two Quantities*.

Once the students noticed this pattern, the TR asked them to explain why the area grew by a constantly-changing rate of change that increased by 3 $cm^2$ per each 1-cm increase in height: "Where does the 3 come from? What does the 3 have to do with the picture?" This press for a quantitatively-based justification served as a support to emphasize attention on how the quantities changed together. Daeshim suggested, "1.5 plus 1.5 equals 3":

TR: And where are you getting the 1.5 and the 1.5?

Daeshim: It's, if width increases 1, then length increases 1.5.

Daeshim suspected that the constantly-changing rate of the change of 3 $cm^2$ in area per 1 cm in height was related to the ratio of length to height, 1.5:1, but was not yet sure why.

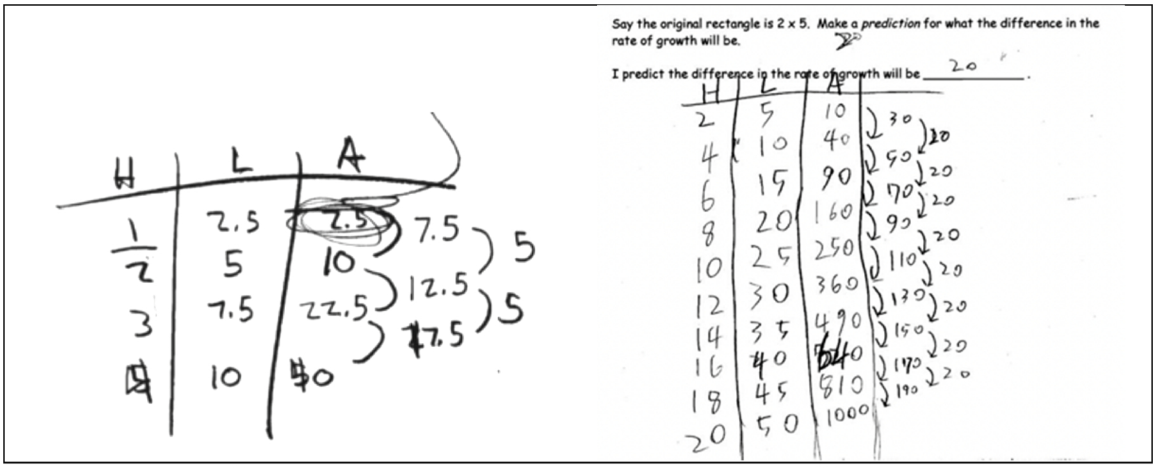
#### 4.2.2. WoU3.2 multiple-unit explicit coordination

Over time it became clear that some students preferred to always increase height values by 1 cm, regardless of the original dimensions of the rectangle. The TR therefore devised a task in which the students had to first predict the changing rate of change for the growth in area of a 2-cm by 5-cm rectangle, and then make their own tables to test their predictions. She suspected that some students might increment the tables by a unit of 2 cm for the height, while others would increment by 1 cm. In addition, by not specifying the amount of increase for the height for which the students' prediction should occur, the TR anticipated that the students would identify different rates of change for the area.

Some students predicted that the changing rate of change for the area would be 5 $cm^2$, others predicted 10 $cm^2$, and others predicted 20 $cm^2$. As anticipated, the students did not initially specify for what change in height their prediction referred to. Additionally, when testing their predictions, some students created tables with a height increase of 1 cm, while others created tables with a height increase of 2 cm (Fig. 9). This difference in prediction led to a disagreement among the students about whether the constantly-changing difference in the change in area should be 5 $cm^2$ or 20 $cm^2$. Jim claimed "It's 5." Bianca exclaimed, "You guys! It's neither! It's, it's, it's 20!" Daeshim also found the constantly-changing difference of change of area to be 20 $cm^2$, which bolstered Bianca's confidence that she was correct and Jim was not.

Bianca: Daeshim is right. And mine is right.

Jim: No, but, I'm doing 1. I'm going by 1.




(a) (b)

**Fig. 9.** (a) Jim's and (b) Daeshim's tables for a 2cm × 5cm growing rectangle prediction task.

Tai: No, but his is going by 1 too.

Jim: *He's* going by 2's. But *I'm* going by 1's.

Jim introduced the need to attend to the change in height values in the students' respective tables. The students then realized that the organization of their tables, particularly the magnitude of height increases, was relevant for determining the constantly-changing rate of change in the area. They concluded that both answers were correct, depending on the choice of height increase. Tai explained, "Yeah, 20 and 5 would both work because we're going up by 2's, and they're going up by 1's." Tai's articulation of a constantly-changing rate of change in area of 20 $cm^2$ per change in height by 2 cm is evidence of *WoU3.2b Explicit Coordination of Second Differences with a Multiple-Unit Change in Another Quantity.*[3] The teacher move of asking the students to create their own tables, and task design feature to predict the constantly-changing rate of change, both supported the students in attending to coordinating changing quantities for magnitude increases other than 1. In addition to creating tables, the TR's continued prompts to create drawings of the quantities in the growing rectangle situation also encouraged an explicit focus on coordinating changes in quantities for multiple-unit iterations.

The students became accustomed to identifying different changes in height as coordinated with changes in length and constantly-changing rates of change in area. The students typically did this as a response to the regular occurrence of needing to reconcile different area rates that emerged as a consequence of different table organizations. As a way to navigate these differences across the students' work, a norm emerged that it was critical to become explicit about the amount of change for each of the relevant quantities.

### *4.2.3. WoU3.3 partial-unit explicit coordination*

A final WoU in *WoT3* entails coordination of partial units (less than 1). In the following example, the TR presented a task comparing height and area in a table, with the magnitude of change for height being 0.5 cm. In attempting to determine the changes in area, the students found the corresponding length values for each height value (Fig. 10a), determining that the length increased by 1.5 cm for every ½-cm increase in height. They also found that the constantly-changing rate of change for the area was 1.5 $cm^2$ for every ½-cm increase in height (Fig. 10b). In Jim's words, "So, like these *two*, it's 1.5 [indicates an increase across two rows, rather than one row]. These two it's 1.5. It's going up 3."

By this point in the TE, the students had developed a norm of stating the constantly-changing rate of change in area for a 1-unit increase in height. Jim realized that claiming 1.5 $cm^2$ was misleading, because that was not for a 1-cm increase, but rather for a ½-cm increase. Thus, he wanted to express the constantly-changing second differences in area as 3 $cm^2$, because he recognized that the increase of 1.5 $cm^2$ was coordinated with ½-cm rather than 1-cm increments in height. We call this *WoU3.3b Explicit Coordination of Second Differences with a Partial-Unit Change in Another Quantity*.

Introducing a table with an increment less than 1 for the height did support the students' partial-unit coordination of height with length and height with changes in area. However, we posit that this instructional support would not have been effective had it not occurred after the establishment of a norm that it was necessary to explicitly attend to the magnitude of height increases. That norm began when the students encountered conflicting results from creating their own tables with different magnitudes of change, and it

[3] We would like to draw attention to a subtle nuance in the language we chose to describe the students' WoU here. First, we describe Daeshim's attention to a "constantly-changing difference of change in area." We then shifted our language to describe Tai's "constantly-changing rate of change in area per height." Instructionally, we aimed to support students' attention to construct constantly-changing rates of change of area with respect to height (and with respect to the ratio of height to length), as described in Section 3.1. However, in analyzing the students' WoT and WoU, we take care to characterize a model of the students' mathematics in our descriptions.

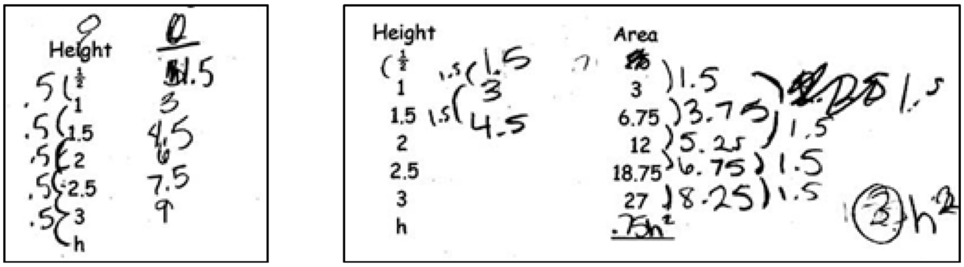


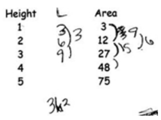
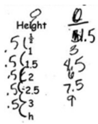
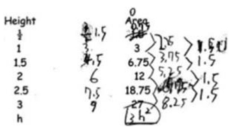

**Fig. 10.** (a) Ally's and (b) Bianca's tables for a growing rectangle growing by ½-cm in height.

continued as the TR deliberately introduced rectangles with configurations that would encourage students to increment height values by amounts other than 1.

Table 4 summarizes the WoU under *WoT3 Explicitly Quantified Coordinated Change*. We found these WoU emerged in response to several instructional supports including: *teacher moves* such as explicitly drawing attention to and identifying quantities in tables,

**Table 4**
WoT3 Explicitly Quantified Coordinated Change, Related WoU, and Instructional Supports.

*WoT3 Explicitly Quantified Coordinated Change* (Conceiving of multi-quantity variation with explicit quantification of both quantities)

| WoU | Definition | Data Example | Instructional Support |
|---|---|---|---|
| *WoU3.1a Single-Unit Explicit Coordination of Change in Two Quantities* | Student coordinates the change in two or more quantities together, and also quantifies the amount of both changes. In this case, change in one quantity is a unit change. | Jim: "The way it kinda looks. Like, it's going up 1 over 2 every time. So it looks like it." | *Teacher Moves:*<br>• Explicitly draw attention to and identify quantities;<br>• Ask students to create their own tables of two or more quantities changing together;<br>• Press for quantitatively-based justifications; and<br>• encourage students to create drawings or diagrams and name quantities. |
| *WoU3.1b Explicit Coordination of Second Differences with a Single-Unit Change in Another Quantity* | A single-unit change in one quantity is coordinated with the difference in the rate of growth of the area. | Tai: "Yeah, 20 and 5 would both work (for second difference in area) because we're going up by 2 s (in height) and they're going up by 1 s (in height)." *Note: this is also an example of WoU3.2b.* | |
| *WoU3.2a Multiple-Unit Explicit Coordination of Two Quantities* | Student coordinates the amount of change for two quantities for multiple-unit changes in one of the quantities. | Bianca: "Each time, it's going up by eight [in length], and each of these [height] is going up by two. The ratio is two to eight, or one to four." | *Teacher Moves:*<br>• Prompt students to create their own tables of two or more quantities changing together; and<br>• Encourage students to create drawings or diagrams and to name quantities.<br>*Task Design Feature:*<br>• Predict the changing rate of change from a table of quantities representing quadratic growth.<br>*Norm:*<br>• Be explicit about the amount of change for each of the relevant quantities. |
| *WoU3.2b Explicit Coordination of Second Differences with a Multiple-Unit Change in Another Quantity* | A multiple-unit change in one quantity is coordinated with the difference in the rate of growth of the area. | Bianca: You add 3 to the length and 6 to the area each time.<br> | |
| *WoU3.3a. Partial-Unit Explicit Coordination of Two Quantities* | Student coordinates the amount of change for two quantities for partial-unit change in one of the quantities. | Ally's table:<br> | *Teacher Moves:*<br>• Ask students to create their own tables of two or more quantities changing together; and<br>• Encourage students to create drawings, diagrams, or graphs.<br>*Norm:*<br>• Be explicit about the amount of change for each of the relevant quantities.<br>*Task Design Feature*:<br>• Investigate a quadratic growth situation where change in independent quantity (Δx) is less than 1. |
| *WoU3.3b. Explicit Coordination of Second Differences with a Partial-Unit Change in Another Quantity* | A partial-unit change in one quantity is coordinated with the difference in the rate of growth of the area. | Daeshim's table:<br> | |

drawings, or diagrams students created; *norms* such as being explicit about the amount of change for each of the relevant quantities; and *task design features*. Across these ways of understanding, students quantified the change in two or more linked quantities (*WoU3.1a*, *WoU3.2a*, *WoU3.3a*), and explicitly quantified coordination of second differences in the dependent variable with another quantity (*WoU3.1b*, *WoU3.2b*, *WoU3.3b*). We found the instructional supports for *WoU3.1a* and *WoU3.1b* were the same (likewise for *WoU3.2a* and *WoU3.2b*; and *WoU3.3a* and *WoU3.3b*).

### 4.3. Transition III: establishing dependency relations of change

The third transition occurred when the students began to articulate *WoT4 Dependency Relations of Change* relating height, length, and area. Dependency relations of change means that the students recognized and identified how changes in linked quantities were dependent on other quantities or changes in those quantities. *WoT4* includes three WoU: (a) *Recognition that Change in One Quantity Determines Change in the Other* (*WoU4.1*), (b) *Identification* of *How Change in One Quantity Determines Change in the Other* (*WoU4.2*), *and* (c) *Translation of Dependency Relations of Change to Correspondence Rules* (*WoU4.3*). We found nine instructional supports that fostered these WoU. The *teacher moves* included: (1) pressing for quantitatively-based justifications. The *norms* included: (2) being explicit about the amount of change in relevant quantities, (3) students responding to and building on one another's ideas, and (4) having multiple student responses shared for the same task. *Tasks* were designed to include: (5) quadratic growth situations that allow for multiple different ways of interpreting and explaining one's reasoning, (6) prediction of the constantly-changing rate of change in the dependent quantity from a table of quantities, (7) investigation of a quadratic growth situation given an initial independent quantity ($x$) greater than 1, (8) investigation of a collection of tables for quadratic growth where the change in the independent quantity is greater than 1 ($\Delta x > 1$), and varied across the set, and (9) making conjectures about generalized relationships among linked quantities for a quadratic growth context. In the following sections we provide excerpts to illustrate how these instructional supports encouraged the development of the three WoU.

#### 4.3.1. WoU4.1 recognition that change in one quantity determines change in the other

The TR had been engendering norms in the classroom in which multiple student responses were elicited for a single question or task, and students were encouraged to respond and build on each other's ideas. Moreover, tasks allowed multiple different ways of interpreting and explaining one's reasoning. In this context, the students started to recognize and identify how changes in one quantity determined changes in the other. In one example, the students were pressed to give a quantitatively-based argument for why the constantly-changing rate of change of the area for a 1 cm by 2 cm rectangle was 4 $cm^2$. Samantha's table and diagram are shown

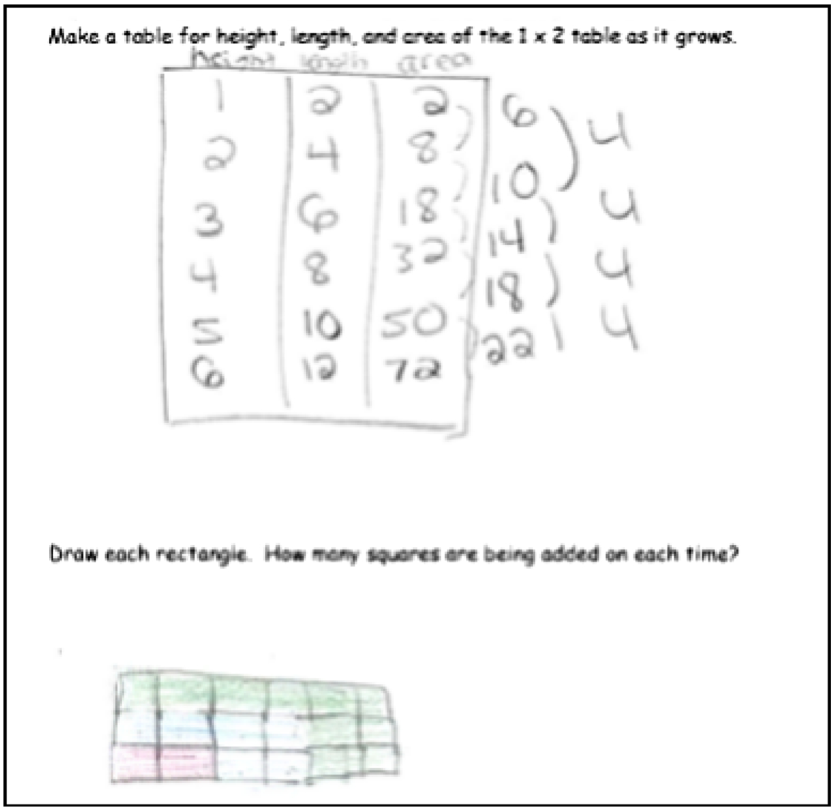



**Fig. 11.** Samantha's work on the 1 × 2 growing rectangle task.

in Fig. 11.

The TR pressed, “So, any idea why it’s 4, instead of 2 or 5 or something?” Jim explained, “When the dimensions of the box change, then the you have your rate of rate of growth of the rate of growth are different.” The TR continued to press for students to connect their reasoning to the quantities height and length:

TR: What does it have to do with the dimensions of the rectangle? In other words, what's the 4 got to do with the dimensions of my original rectangle which I believe was uh…1 to 2? Bianca?

Bianca: “Well, it like…it has to do with the ratio, but I really can't explain it.”

In this exchange, Jim explained that the constantly-changing rate of change depended on the dimensions of the rectangle, but he was not exactly sure what that relationship was. Bianca built on Jim’s idea to conjecture the it was related to the ratio (of the height to length), but could not articulate how exactly the ratio affected the difference in the rate of growth of area. In a later exchange, Jim expressed, “So your rate of growth (for the area) can change no matter what.”

The students understood that the quantities height and length affected the constantly-changing rate of change of the area, but could not yet identify precisely how those quantities, or changes in them, affected the change in area. We take this as evidence of *WoU4.1 Recognition that Change in One Quantity Determines Change in the Other*. The instructional goals were focused on helping students recognize and name relationships among linked, changing quantities. Tasks were designed to support students to see a dependency relationship between changing quantities with explicit attention to identifying those quantities in both pictures of the growing rectangle and tables of values.

### *4.3.2. WoU4.2 identification of how change in one quantity determines change in the other*

Eventually the students were able to transition to not only understanding *that* changes in one quantity determine changes in another, but they could also identify *how* this occurred, particularly in terms of how changes in height and/or length affected the constantly-changing rate of change of area. For instance, for the Four Tables task (see Fig. 12), Tai found a way to relate the constantly-changing rate of change of the area with respect to the changes in both the height and the length. Tai said: “Take the [difference in the change] of the area, and you divide by the difference in the length…and also divided by, the difference in height…and, always equals 2.” Tai identified a dependency relation between ΔH, ΔL, and ΔΔA; he verbally expressed the relationship $\left[\frac{\frac{\Delta\Delta A}{\Delta L}}{\Delta H}\right]$ = 2.]. Another student, Bianca, noticed that ΔΔA could be expressed in terms of change in height values: “So it’s basically like 3 [times difference in height] squared times 2.” Tai expressed this finding as the equation: 3•[difference in height]$^2$ = [constantly-changing difference in area].” We call this *WoU4.2 Identification of How Change in One Quantity Determines Change in the Other*. In this WoU, students articulated how changes in a quantity such as height or length determined changes in the change in Area, for example.

We observed the *WoU4.2* to emerge in the context of deliberate norms, teacher moves, and task design principles. The students approached the Four Tables task by making changes in linked quantities explicit in the table representation, which had become an established norm. Prior to the Four Tables task, the class had investigated other tasks that encouraged prediction of the difference in the rate of growth. In one task, students predicted the constantly-changing rate of change in a rectangle’s area given only the ratio of its height to length (recall Fig. 9b, for example). These tasks led the class to explore several student conjectures, each of which demanded a quantitatively-based justification. We surmise that this repeated reasoning of making and testing predictions cultivated a norm that encouraged the expression of dependency relationships. Finally, the tasks were designed to: (a) investigate a quadratic growth situation given an initial independent quantity greater than 1, (b) investigate a collection of tables with varied changes in the independent quantity, and (c) elicit conjectures about generalized relationships among linked quantities for a quadratic growth context. The sequencing of tasks of the same type (such as the prediction tasks and the Four Tables task) encouraged the students to

Here are 4 tables for the height versus the **area** of the exact same rectangle that is growing in proportion

| Height | Area | Height | Area |
|---|---|---|---|
| 1 | 3 | 2 | 12 |
| 2 | 12 | 4 | 48 |
| 3 | 27 | 6 | 108 |
| 4 | 48 | 8 | 192 |
| 5 | 75 | 10 | 300 |

| Height | Area | Height | Area |
|---|---|---|---|
| 5 | 75 | 10 | 300 |
| 10 | 300 | 20 | 1,200 |
| 15 | 675 | 30 | 2,700 |
| 20 | 1,200 | 40 | 4,800 |
| 25 | 1,875 | 50 | 7,500 |

In each case, what does the formula for area have to do with the constantly-changing rate of change of the area?

**Fig. 12.** The Four Tables Task.

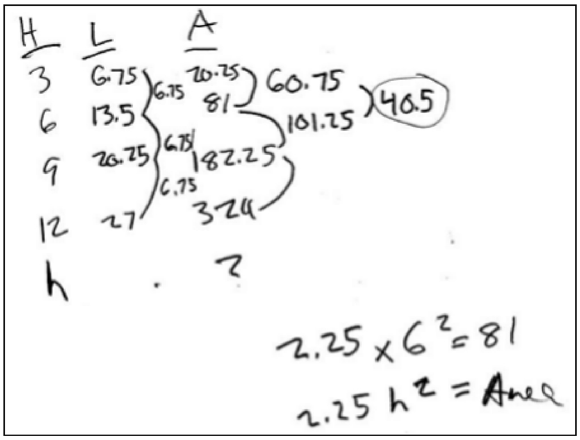



**Fig. 13.** Jim's table and equation relating height and area of a growing rectangle.

engage in repeated reasoning to test and/or formalize conjectures about generalized relationships. We call attention to these generalizing activities because ultimately, this kind of generalizing also helped the students develop quadratic equations, as we elaborate next in the final WoU.

#### *4.3.3. WoU4.3 translations of dependency relations of change to correspondence rules*

The *WoU4.3 Translations of Dependency Relations of Change to Correspondence Rules* emerged when the students leveraged their coordination thinking to create and make sense of correspondence rules. At this point in the TE, many instructional supports had become normative. The students repeatedly engaged in reasoning that led them to identify and quantify dependency relations of change. Further, the TR implemented carefully sequenced sets of tasks aimed at prompting students to link their understandings of dependency relations of change to a symbolic rule. For example, the TR set up a task asking students to write an equation to: (a) find the area of the rectangle when the height is h, and (b) relate the equation to the constantly-changing rate of change of area with respect to height. Consider Jim's work on this task, shown in Fig. 13.

The students hypothesized that the difference in the rectangle's length divided by the difference in the height was the coefficient $a = \frac{\textit{difference in length}}{\textit{difference in height}}$ for the symbolic rule $Area = ah^2$. The students explained:

Jim: 6.75 divided by three equals 2.25, so if I did 2.25 times six squared equals 81. It works. 2.25 times $h^2$ equals area.

TR: So what works?

Daeshim: Um, dividing by um, dividing by how many are going up by each time. So, like the, what's in the length, divided, divided by how much [inaudible] by height is equal to.[4]

In this exchange, the students expressed the relation between changes in two quantities as an algebraic rule. The task was designed to have students give a far prediction for the area and to elicit a generalization of the relationship between the height and the area. The students were accustomed to linking quantities as relationships of explicit coordinated change, and making and testing predictions about generalized relationships. The TR made these normative practices explicit by asking the students to test their hypothesis and to state a generalized rule. When students stated their rules, the TR then pressed for quantitatively-based justifications: "So, it's this difference in the length divided by three, and why divided by three?" to which the students gave a quantitatively-based answer "Because that's what you're going up by each time." A summary of the WoU within the *WoT4 Dependency Relations of Change* is given in Table 5.

### *4.4. Transition IV: establishing correspondence*

In the final transition to *WoT5 Correspondence*, we found one main WoU: *Correspondence Relations Between Independent and Dependent Quantities* (*WoU5.1a, WoU5.1b*). We identified nine instructional supports: *Teacher Moves* (1) Ask students to create an algebraic rule that relates an independent and dependent quantity; (2) Press for quantitatively-based justifications; (3) Ask students to generalize a mathematical relationship they identified; *Task Design Features* (4) Investigate a quadratic growth situation where change in independent quantity is greater than 1; (5) Make far predictions of independent and dependent quantities for a given quadratic growth pattern; (6) Elicit generalizations of relationships between independent and dependent quantities using variables; (7) Sequence tasks of the same type to allow students to test conjectures about generalized relationships; and *Norms* (8) Be explicit about the magnitude of change for each of the relevant quantities; and (9) Make and test predictions about generalized relationships

[4] In reading Jim's work in Fig. 13, note that the generalization that $a = \frac{\textit{difference in length}}{\textit{difference in height}}$ was not written down, but rather expressed verbally by both Jim and Daeshim.

**Table 5**
WoT4 Dependency Relations of Change, Related WoU, and Instructional Supports.

*WoT4 Dependency Relations of Change* (conceiving of the change in one quantity as dependent on other quantities or change in other quantities)

| WoU | Definition | Data Example | Instructional Support |
|---|---|---|---|
| *WoU4.1 Recognition that Change in One Quantity Determines Change in the Other* | The magnitude of the change of one quantity, such as height, determines the amount of change in another quantity, such as area (or the difference in the rate of growth of area). Student understands that there is a dependency relation *without* determining what that relation is. | Jim: "When the dimensions of the box change, then the you have your rate of rate of growth of the rate of growth are different." | *Norms:*<br>• Attend to linked quantities as relationships of explicit coordinated change;<br>• Create tables and drawings to express relationships in quadratic growth contexts;<br>• Multiple student responses are shared for a single question or task; and<br>• Students respond to and build on each other's ideas.<br>*Task Design Feature:*<br>• Quadratic growth task allows for multiple different ways of interpreting and explaining one's reasoning.<br>*Teacher Move:*<br>• Press for quantitatively-based justifications. |
| *WoU4.2 Identification of How Change in One Quantity Determines Change in the Other* | The student quantifies the relation between change in one quantity and the difference in the rate of growth of change in the other quantity (area). | Tai: "(Referring to a table of values coordinating successive height ($H$), length ($L$), and area ($A$) values for a growing rectangle) [constantly-changing difference of change] divided by [difference in length] equals…and, divided by [difference in height] equals 2. [Here he verbally expressed the relationship $\left[\frac{\frac{\Delta\Delta A}{\Delta L}}{\Delta H}\right] = 2$.]" | *Task Design Features*:<br>• Investigate a quadratic growth situation given an initial independent quantity ($x$) greater than 1;<br>• Investigate a collection of tables for quadratic growth where the change in the independent quantity ($\Delta x$) is greater than 1 and varied across the set;<br>• Elicit conjectures about generalized relationships among linked quantities for a quadratic growth context; and<br>• Sequence tasks of the same type to allow students to engage in repeated reasoning to test and/or formalize conjectures about generalized relationships.<br>*Teacher Move*:<br>• Press for quantitatively-based justifications.<br>*Norm:*<br>• Make and test predictions about generalized relationships among linked quantities; and<br>• Attend to linked quantities as relationships of explicit coordinated change. |
| *WoU4.3 Translation of Dependency Relations of Change to Correspondence Rules* | Expressing the relation between changes in two quantities in an algebraic rule, such as $y = ax^2$. Student understands the rule in terms of a relation of change. | Jim: "look…[the constantly-changing difference of change] divided by 2 is [the difference in the length]"<br>Bianca: "[the constantly changing rate of change] over 2 equals [the difference in the length]."<br>TR: "What does [the difference in length] have to do with area?"<br>Bianca: "Distance, length. Distance times length."<br>Jim: "[difference in length] is, um, this. The 3$h$ squared." | *Task Design Feature*:<br>• Sequence tasks of the same type to allow students to test conjectures about generalized relationships.<br>• Investigate a quadratic growth situation where change in independent quantity ($\Delta x$) is greater than 1; and<br>• Elicit conjectures about generalized relationships among linked quantities for a quadratic growth context.<br>*Norm*:<br>• Attend to linked quantities as relationships of explicit coordinated change; and<br>• Make and test predictions about generalized relationships among linked quantities.<br>*Teacher Moves:*<br>• Press for quantitatively-based justifications; and<br>• Ask students to generalize a mathematical relationship they identified. |

among linked quantities.

#### 4.4.1. WoU5.1 correspondence relations between independent and dependent quantities

For each correspondence rule that the students wrote, the TR asked them to explain what each variable meant. This press for quantitatively based explanations supported the students to write rules that were grounded in quantitative meaning. For example, Bianca developed a relationship between the side of a growing square and the area of a growing square as "$a = s^2$" or "the area = the side measurement squared". The TR prompted Bianca to further connect this rule to the relevant quantities.

TR: Area equals side squared [writes on the board]. So can you give me an example of that. Like, for instance, this point (4,16)—

Bianca: The 4 is the…4 which is the height and the 4 which is the width [gestures to the height and width of the projected square].

TR: Ok. So the 4 is the height and the length.

Bianca: Yeah.

TR: Where's the 16 in the picture?

Bianca: It's the number of squares inside.

In this example, Bianca articulated both a general correspondence rule relating independent and dependent quantities, $a = s^2$ and a specific instance, of how 4 cm and 4 cm are the side lengths and 16 $cm^2$ is the area, which is why (4,16) works for the rule $4^2 = 16$. In related tasks, the TR asked far prediction questions for very large height values. For instance, given a table of height / area values for a 2 cm by 9 cm rectangle with height values ranging from 2 cm to 8 cm, the TR asked the students to determine the area for a height of 82:

TR: Yeah, so you found out the area when the height's 82. You found it to be 30,258. What if height's *n*?

Jim: Oh, I think I got it: *n* times 4.5 times *n* equals area. Simple. I think.

In Far Prediction tasks, the TR asked students to generalize mathematical relationships by giving prompts such as "what would [the area] be if the height is just *h*?" Sometimes, those prompts were also accompanied by a prompt to write an algebraic rule. The normative practices of being explicit about the amount of change for each of the relevant quantities, as well as making and testing predictions about generalized relationships, remained instructional supports in *WoT5 Correspondence*. The students' engagement in repeated reasoning through a sequence of tasks of the same type was also evidence of an instructional support. A summary of these examples with related definitions is given in Table 6.

## 5. Discussion and conclusion

In this research we sought to better understand how one can engender and model conceptual change in students' WoT and WoU quadratic growth. The students who participated in this study went through four major transitions between five WoT: *Variation*, *Early Coordinated Change*, *Explicitly Quantified Coordinated Change*, *Dependency Relations of Change*, and *Correspondence*. These transitions did not occur spontaneously. We deliberately use the word transition to convey a broader interaction modeling changes in both students' WoU and WoT, and changes in the instructional supports and context vis-à-vis teacher moves, norms co-developed among the TR and the students, and task design features.

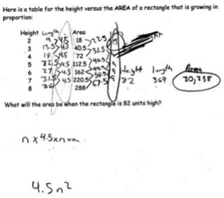
Our findings suggest that meaningful learning of quadratic growth is conceptually challenging for middle-grades students, but possible to attain with tailored instructional supports. A rate of change approach to quadratic growth, coupled with design principles grounded in DNR-based instruction, supported the teaching and learning of quantitative reasoning, representational fluency, and generalization in a dynamic, visualizable context. The learning trajectory we developed offers insight into a model of students' mathematics, and how instruction might foster a dynamic understanding of function, which is critical for success in higher-level mathematics.

### 5.1. Instructional supports for students' WoT and WoU quadratic growth

Although learning and teaching function in conceptually oriented ways remains central in secondary mathematics education, there remain well-documented challenges for both students and teachers (e.g., Wilkie, 2019). In light of the promise of instructional tasks with an emphasis on a rate of change perspective, there remains a need for additional evidence that links students' meaningful learning with effective instructional supports. This study addressed this gap by elaborating types of instructional supports for students' WoT and WoU about quadratic growth. From the design phase of this research, the instructional supports, particularly the tasks, were deliberately engineered to foster and, ultimately, require students to explicitly identify the manner in which two quantities changed together. Our instructional supports were grounded in a theoretical orientation toward the importance of fostering students' quantitative reasoning, and a mathematical goal of supporting students' understanding of quadratic growth as a relationship between co-varying quantities in which one quantity changes at a constantly-changing rate of change with respect to the other. Our conceptually oriented mathematical goals remained central to our design, yet did not overshadow the importance of building models of students' mathematics as the TR and students interacted around a purposeful instructional task sequence.

These design decisions, and a DNR-based approach to instruction, certainly background our findings. However, our categorization

**Table 6**
WoT5 Correspondence, Related WoU, and Instructional Supports.

| *WoT5 Correspondence* (Conceives of direct correspondence relation between independent quantity and dependent quantity) | | | |
|---|---|---|---|
| WoU | Definition | Data Example | Instructional Support |
| *WoU5.1a Specific Correspondence Between Independent and Dependent Quantities* | Student understands a correspondence relation between independent quantity x (height or length) and dependent quantities y (area) for a specific value or values of a relationship. | TR: "Area equals side squared [writes on the board]. So can you give me an example of that. Like, for instance, this point (4,16)—"<br>Bianca: "The 4 is the…4 which is the height and the 4 which is the width [gestures to the height and width of the projected square]."<br>TR: "Ok. So the 4 is the height and the length."<br>Bianca: "Yeah."<br>TR: "Where's the 16 in the picture?"<br>Bianca: "It's the number of squares inside." | *Teacher Moves*:<br>● Ask students to create an algebraic rule that relates an independent and dependent quantity;<br>● Press for quantitatively-based justifications; and<br>● Ask students to generalize a mathematical relationship they identified. |
| *WoU5.1b. Generalized Correspondence Rule* | Student understands the function rule as a generalized instance of a mapping between independent and dependent quantities. Student expresses a correspondence rule as a direct relationship between an independent quantity (height or length) and dependent quantity (area). This may be expressed in words, in a diagram, or in algebraic symbols (e.g., $y = mx^2$ or $A = ah^2$). | TR: "Yeah, so you found out the area when the height's 82. You found it to be 30,258. What if height's n?"<br>Jim: "Oh, I think I got it. n times 4.5 times n equals area. Simple. I think."<br> | *Task Design Features*:<br>● Investigate a quadratic growth situation where change in independent quantity (Δx) is greater than 1;<br>● Make far predictions of independent and dependent quantities for a given quadratic growth pattern;<br>● Elicit generalizations of relationships between independent and dependent quantities using variables; and<br>● Sequence tasks of the same type to allow students to test conjectures about generalized relationships.<br>*Norms*:<br>● Be explicit about the magnitude of change for each of the relevant quantities; and<br>● Make and test predictions about generalized relationships among linked quantities. |

of instructional supports into teacher moves, norms, and task design features was emergent. We found the teacher moves and norms to explicitly foster a discourse community that focused on quantitative reasoning, representational fluency, and generalization. We found the task design features to fall along related categories of repeated reasoning, far prediction and generalization, and doing mathematics.

The set of instructional supports introduced in this paper offers a new framework for elaborating some of the mechanisms that can support students' conceptual change. We found that the tasks and task design features, teacher moves, and norms interactively worked together to support transitions in students' WoT and WoU. An example of the integrated nature of instructional supports was given in Transition II; the teacher moves and task design features did not work in isolation, but rather in concert with carefully developed norms such as being explicit about the magnitude of change in linked quantities. For instance, introducing tables incremented by units less than 1 cm in height encouraged the development of a partial-unit coordination of changes in height values with changes in area values. These tables, however, would likely not have been effective in supporting this coordination had the students not already become accustomed to explicitly identifying the change in height. This was a norm that established slowly over the course of multiple task sequences, and it was only once this norm was firmly in place that the students were able to begin explicitly coordinating changes for increments less than 1. Similarly, a number of the associated teacher moves, such as pressing for quantitatively-based justifications, were made possible and effective in the context of tasks situated in a quantitatively-rich situation and norms encouraging students to build on one another's ideas. It is the interaction between task design, teacher moves, and norms that enable transitions in students' WoT and WoU; tasks alone cannot carry the full responsibility for enacting conceptual change.

### 5.2. Quantitative reasoning about functions and DNR

We found that the students' WoT about quadratic growth (*WoT1* to *WoT5*) developed in reflexive relation with more specific WoU within the particular content of quadratic growth. Recall the left-most column of Table 1, and the left-hand side of Fig. 4. This research extends current characterizations of students' conceptual learning of quadratic function (e.g., Ellis, 2011a, 2011b; Wilkie, 2019), and taxonomies for conceptual learning goals (e.g., Lobato et al., 2012) by elaborating the nature of transitions among

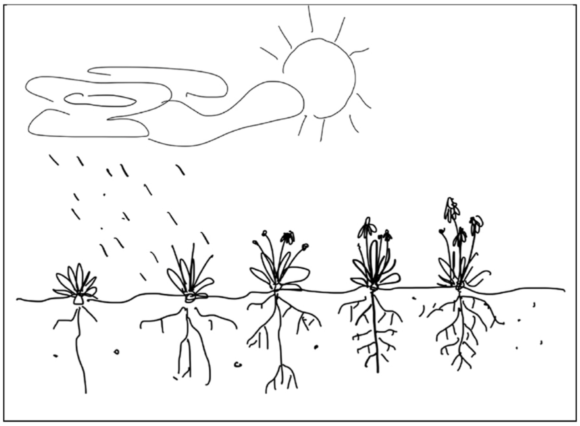


**Fig. 14.** A visual metaphor for a learning trajectory of change in models of students' mathematics (flower) fostered by a system of interacting instructional supports (sun, rain, soil).

qualitatively distinct ways of thinking. In particular, this research contributes to a growing body of work on learning trajectories that explicitly attends to the duality of students' broader ways of thinking about mathematics content and specific ways of understanding mathematical ideas (Empson, 2011).

This research also provides nuance into the kinds of WoT that are involved in reasoning about quadratic function, expanding on Harel's (2013) framing of the kinds of broader conceptual tools students bring to bear in a specific learning context. *Early Coordinated Change* and *Dependency Relations of Change*, for example, extend current framings of quantitative reasoning about functions, and quadratic functions in particular. In common characterizations of students' quantitative reasoning about functions, much attention is given to explicit coordinated change (cf. *WoT3*; see also Confrey & Smith, 1994) or covariation (Castillo-Garsow, 2013; Saldanha & Thompson, 1998), and correspondence (cf. *WoT5*) reasoning about functions (see also Smith, 2003).

### 5.3. Visualizing learning trajectories

In writing about the quadratic growth learning trajectory as a series of transitions between WoT and instructional supports we do not intend to convey a linear progression of ideas, nor do we intend to convey a neat structuring of developmental stages that occur in sequence. To illustrate an example of how learning is not linear, *WoT5 Correspondence* tended to co-occur with *WoT4 Dependency Relations of Change* (this co-occurrence is illustrated in Fig. 4). We choose to introduce *WoT5* last in Transition IV because it tended to occur late in students' thinking, and it was instructionally last in the sequence. Different students will inevitably traverse a learning trajectory in different ways. Our construction of a learning trajectory in this paper relied on our best model from the data as guided by our theoretical orientation. We do not claim this is the only learning trajectory for quadratic growth.

We find the metaphor of growing flowers in their appropriate habitat to be a useful tool for thinking about the interrelationships among our theory-driven design, characterizations of students' WoT and WoU, and articulation of instructional supports as a set of teacher moves, task design features, and norms. In Fig. 14 we introduce a Visual Metaphor depicting transitions in both the growth of the plant (change in the mathematics of students), as well as an interacting system of soil, water, and sun as a nourishing environment engendering growth (instructional supports). This metaphor affords flexibility in conceptualizing both the interaction of a growing flower with the environment, as well as the system of environmental interactions. For learning trajectories, this visual metaphor makes these two features salient: (a) how the child interacts with the learning environment, and (b) how the tasks and task design features, teacher moves, and norms interactively work together to support children's learning.

Often learning trajectories positing models of students' mathematics are taken to be "just a bunch of flowers" without due attention to the connectedness of how flowers grow in response to the environment nurturing them. With this visual metaphor, just as how different ecological environments nurture different kinds of plant growth, we can imagine how different kinds of learning environments and instruction might support qualitative differences in children's learning.[5] The learning trajectory developed in this study contributes evidence of how change in students' WoU and WoT can be engendered by theory-driven instructional supports. More broadly, the paper contributes an example of how learning trajectories research can attend to both the nature of students' learning and the nature of supports for learning (beyond a sequence of tasks).

[5] We thank an anonymous reviewer for pointing out that certain plants are not suited to grow in certain climates. If a plant does not grow in a certain environment, the problem is not with the plant. Likewise, if a child does not learn in a particular instructional setting, the problem is not with the child. We do not intend for the visual metaphor of a learning trajectory to convey a deficit orientation toward children. Rather, the metaphor highlights the importance of both understanding and supporting the richness of students' mathematics as contextualized in learning environments.

### 5.4. Concluding remarks

Theory development that links theory of learning and theory of instruction is needed (Simon et al., 2010). We see this study as an example of research that deliberately attends to both learning and teaching to explain change in students' WoU and WoT about mathematics. Our learning trajectory is a networking of theories of learning and theories of instruction that are grounded in empirical observation of students' mathematical activity. It is a nuanced model linking conceptual learning goals, models of students' WoT and WoU, and instructional supports that can engender students' transitions from one WoT to another. True to developmental research (Brown, 1992; Gravemeijer, 1994) our theoretical orientation guided both the engineering of the learning and teaching context and our analysis of data. For example, the TR was active in hypothesizing students' activity, and chose to enact instructional supports that extended students' reasoning, effectively engendering a shift in students' WoT. We see great potential in expanding research and practice around the use and creation of learning trajectories that link these inter-related dimensions of teaching and learning. Learning trajectories research can help propel the field by unifying research programs that seek to understand what students' conceptual learning of mathematical ideas and WoT can look like, and how instruction can support that learning.

## CRediT authorship contribution statement

**Nicole L. Fonger:** Conceptualization, Formal analysis, Visualization, Writing - original draft, Writing - review & editing. **Amy B. Ellis:** Conceptualization, Formal analysis, Funding acquisition, Writing - original draft, Writing - review & editing. **Muhammed F. Dogan:** Data curation, Formal analysis, Writing - review & editing.

## Appendix A. Supplementary data

Supplementary material related to this article can be found, in the online version, at doi:https://doi.org/10.1016/j.jmathb.2020.100795.

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