

## TRANSFER STRAND

---

# A Taxonomy for Categorizing Generalizations: Generalizing Actions and Reflection Generalizations

Amy B. Ellis

*Department of Curriculum and Instruction  
University of Wisconsin–Madison*

This article presents a cohesive, empirically grounded categorization system differentiating the types of generalizations students constructed when reasoning mathematically. The generalization taxonomy developed out of an empirical study conducted during a 3-week teaching experiment and a series of individual interviews. Qualitative analysis of data from teaching sessions with 7 seventh-graders and individual interviews with 7 eighth-graders resulted in a taxonomy that distinguishes between students' activity as they generalize, or *generalizing actions*, and students' final statements of generalization, or *reflection generalizations*. The three major generalizing action categories that emerged from analysis are (a) relating, in which one forms an association between two or more problems or objects, (b) searching, in which one repeats an action to locate an element of similarity, and (c) extending, in which one expands a pattern or relation into a more general structure. Reflection generalizations took the form of identifications or statements, definitions, and the influence of prior ideas or strategies. By locating generalization within the learner's viewpoint, the taxonomy moves beyond casting it as an activity at which students either fail or succeed to allow researchers to identify what students see as general, and how they engage in the act of generalizing.

One of the primary aims of educational practice is to help students develop robust, generalizable knowledge that will support their abilities to create generalizations

in the classroom as well as transfer their knowledge to new settings and conditions. Two major traditions that have addressed the generalization of learning are the fields of generalization research in mathematics education and transfer research in education, psychology, and cognitive science. While generalization research typically examines the creation of a mathematical rule or property (Carpenter & Franke, 2001; English & Warren, 1995; Lee, 1996), transfer research has historically studied the application of knowledge learned in one situation to another situation (described by Singley & Anderson, 1989). Although these approaches might appear different at first glance, this article identifies a number of connections between the conceptions, theoretical orientations, and methodologies common to both research traditions.

The creation of categorization systems has been an important component of research in both approaches to the generalization of learning. For example, transfer researchers have created categorization schemes to explain experimental findings demonstrating low incidences of transfer (Barnett & Ceci, 2002; Perkins & Salomon, 1988). Classifying types of generalizations in mathematics education research has led to a greater understanding of the different ways in which students construct general rules to fit particular cases or data (Garcia-Cruz & Martinon, 1997; Lannin, 2003; Stacey, 1989). However, traditional research studies in both fields tend to rely on expert models of performance, which could result in a failure to capture instances that may constitute generalization from the learner's perspective. Accordingly, many studies ultimately categorize tasks and correct mathematical strategies rather than learners' evolving mental processes.

The purpose of this article is to create a taxonomy for generalization that extends previous work by: a) operating from a student-oriented perspective, b) making connections between the transfer literature and generalization research in mathematics education, and c) presenting a system developed from methods informed by a reconceptualization of what it means to generalize. In doing so, the generalization taxonomy represents a coherent, empirically grounded categorization system that distinguishes processes and results of generalizing activity across multiple interconnected dimensions. The connections identified in the taxonomy offer a way to identify evolving levels of sophistication in generalizing activity.

## THEORETICAL FRAMEWORK: RECONCEIVED MODELS OF GENERALIZATION AND TRANSFER

### One Problem in Two Research Traditions: Reliance on the Observer's Perspective

Classical transfer studies privilege the perspective of the observer because the researcher predetermines what counts as transfer based on the correspondences that

experts make between initial learning and transfer tasks (Lobato, 2006). As a result, transfer studies often reveal more information about the researcher's construction of similarity than the learner's (Pea, 1989). Although transfer researchers report an interest in examining the influence of prior learning experiences on attempts to solve problems in new situations (Reed, Ernst, & Banerji, 1974), Lobato (in press) claims that in practice, classical transfer experiments examine the formation of highly valued expert generalizations rather than the generalization of learning more broadly. In addition, the requirement that researchers find improved performance between transfer tasks may prevent them from capturing instances in which students construe situations as similar, but do not benefit from such constructions in terms of increased performance (Lobato, 2003). This could result in traditional studies underreporting the generalization of learning.

A reliance on an observer's perspective results in similar limitations with studies examining students' generalizations within the mathematics education community. Generalization is viewed as not only the production of a correct mathematical rule or principle, but also as the production of a rule that researchers predetermine to be mathematically useful within a particular context (Orton & Orton, 1994; Stacey & MacGregor, 1997). Although students may perceive a number of different patterns and properties, their focus on those which researchers do not deem relevant is not typically taken as an appropriate generalization. As a result, studies examining students' generalizations often report students' difficulties in recognizing, using, and creating general statements (English & Warren, 1995; Kieran, 1992; Lee, 1996; Stacey & MacGregor, 1997). Because work on generalization predetermines what type of knowledge counts as general, it may fail to capture instances in which students may perceive a common element across cases, extend an idea to incorporate a larger range of phenomena, or produce a general description of a phenomenon, regardless of its correctness.

### Classical Categorization Systems

The use of an observer's perspective has influenced the nature of the categorization systems that have been used to distinguish between different types of transfer from a classical perspective and different types of generalization from the mathematics education tradition. For instance, Gagné (1977) distinguished between *lateral transfer*, the generalization of what is learned in one situation to a new situation at roughly the same level of complexity, and *vertical transfer*, the learning of lower level skills or knowledge as facilitating the acquisition of more complex skills or knowledge. Another typical distinction is between *near* and *far transfer* (Detterman, 1993). Near transfer refers to cases in which the original learning situation and new situation are quite similar; far transfer addresses cases in which the two situations are quite different. Another major distinction of transfer types is that of *specific* versus *nonspecific* or *general transfer* (Singley & Anderson, 1989). In

specific transfer, the learner transfers domain specific content knowledge, whereas in general transfer, the learner transfers principles or heuristics to new situations. In each of these categorization systems, the observer's perspective informs what constitutes similar levels of complexity, similar situations, or domain-specific knowledge.

Many parallels exist in generalization research in mathematics education: for instance, Stacey (1989) describes the difference between near generalization, a description of a pattern allowing one to determine the next term in a sequence, and far generalization, the construction of a general rule. Davydov (1990) and Krutetskii (1976) differentiate between subsuming particular cases into a general concept versus developing a general concept from a particular case. Other researchers have gone further by elaborating multiple levels or strategies. For example, Garcia-Cruz and Martinon (1997) developed a framework for three levels of generalization in linear patterns. The levels distinguish between a student's ability to recognize and use the iterative nature of a pattern, establish an invariant from an action performed on a pattern, and generalize a strategy to a new problem. Again, what constitutes distinctions such as near or far rely on the researcher's point of view. In addition, by focusing on correct mathematical strategies, mental acts that cut across strategies may be overlooked and generalizing processes that result in incomplete or incorrect generalizations may be omitted.

### Alternative Approaches to Transfer and Mathematical Generalization: Adopting the Actor-Oriented Perspective

Transfer has been notoriously difficult to demonstrate in the laboratory (Detterman, 1993; Gruber, Law, Mandl, & Renkl, 1996), and mathematics education research demonstrates similar difficulties in capturing generalization (English & Warren, 1995; Lee, 1996; Stacey & MacGregor, 1997). This suggests one of two possibilities: either concept understanding does not typically transcend the context in which it is developed (Brown, Collins, & Duguid, 1989; Perkins & Salomon, 1989), or existing constructs fail to account for all of the ways in which students might be generalizing their knowledge. Given that students' and teachers' everyday experiences demonstrate instances of generalizing on a regular basis, turning to a reconceptualization of transfer could inform work on generalization by expanding the range of student acts that may actually constitute generalizing phenomena.

In response to these limitations, researchers have proposed alternative approaches that characterize transfer more broadly (Beach, 1999; diSessa & Wagner, 2005; Greeno, 1997; Lobato, 2006; Marton, 2006). For instance, Lobato (2006) describes transfer as the influence of learners' prior activities on their activity in novel situations, whereas Marton (2006) addresses transfer in terms of how learning in one situation affects or influences what the learner is capable of doing in an-

other situation, even if the influence results in differential behavior. These alternative approaches characterize transfer as the generalization of learning; thus, they capture a broader range of student actions as transfer, including students' mathematical generalizations.

### *Actor-Oriented Transfer*

In an attempt to address the limitations of observer-oriented transfer research, Lobato (2003, in press-a) developed a framework for transfer called the actor-oriented transfer perspective, in which the researcher shifts from an observer's (or expert's) viewpoint to an actor's (learner's) viewpoint. From this perspective, transfer is the generalization of learning, which can be seen as the influence of a learner's prior activities on his or her activity in novel situations. The actor-oriented perspective seeks to understand the processes by which people connect learning experiences with new situations. This connection-making between situations most predominately involves the process of similarity-making, but it can also involve the processes of discerning differences and modifying situations (Lobato, Clarke, & Ellis, 2005; Lobato & Siebert, 2002). The actor-oriented framework allows the researcher to identify what is salient for students. Attending to students' perceptions of similarity, regardless of their correctness, can then enable the identification of supports for specific types of generalizations.

### *Connections Between Actor-Oriented Transfer and Generalization*

From the actor-oriented perspective, one can identify a number of similarities between processes of transfer and processes of generalization. Mathematics education researchers have described generalization in three broad ways: (a) the development of a rule that serves as a statement about relations or properties (Carpenter & Franke, 2001; English & Warren, 1995; Lee, 1996), (b) the extension or expansion of one's range of reasoning beyond the case or cases considered (Dubinsky, 1991; Harel & Tall, 1991), and (c) the identification of commonalities across cases (Dreyfus, 1991; Kaput, 1999). These characterizations mirror Lobato's reconceived view on several fronts. For instance, the development of a rule and the expansion of one's range of reasoning are both forms of the generalization of learning, whereas the identification of commonalities can be seen as the process of similarity making. The fact that these types of generalizing actions could be captured under the umbrella of the transfer phenomenon suggests that we should look to reconceptions of transfer in an attempt to address some of the limitations of previous research on mathematical generalization.

Moreover, there is a subset of research on generalization in mathematics education which could be interpreted as consistent with reconceived views of transfer, particularly the actor-oriented transfer perspective. For instance, Harel and Tall

(1991) distinguished between expansive generalization, in which one extends his or her scheme without changing it, and reconstructive generalization, in which a scheme undergoes reorganization. Viewed through the actor-oriented transfer lens, one could study students' scheme expansions and reconstructions without being restricted by normative notions of correctness. Similarly, Piaget's distinction between inductive and constructive generalization is compatible with the actor-oriented view (Piaget & Henriques, 1978). Inductive generalization represents the extension of the field of application of an existing mental structure, whereas constructive generalization involves the generation of new structures and contents. Viewed through the actor-oriented lens, one could study students' scheme generations and extensions without first having to determine what constitutes appropriate general knowledge.

However, a limitation of the actor-oriented transfer perspective is that it does not distinguish between types of generalization, instead capturing a wide range of generalizing acts as forms of transfer. Thus, a categorization scheme of types of actor-oriented transfer is needed. The types of generalization identified within the mathematics education literature that are consistent with an actor-oriented approach are promising to use as a starting point, but they fall short of presenting a coherent model that would distinguish multiple dimensions of generalization and articulate relations among types of generalizations. One of the aims of this article is to connect to the reconceived transfer literature to inform work on generalization in mathematics education. By taking a broader notion of what it means to generalize, this study captures a wide range of student actions as they attempt to produce generalizations about linear functions. These actions are presented in the form of a comprehensive, empirically grounded taxonomy, which describes multiple inter-related types of generalizing.

## A Shift in Methods

The data collection methods that informed the generalization taxonomy differed from those typically employed in either classical transfer experiments or generalization studies in mathematics education. In particular, transfer studies typically present students with a learning situation followed by a transfer task. The transfer task is determined to be both separate and distinct from the learning situation, particularly in terms of surface features, but still sharing some structural features in common. What constitutes a surface feature versus a structural feature, or what counts as similar or different situations are determined by the researchers; students' views on these similarities or differences do not inform the design of classical studies. Similarly, many generalization studies in mathematics education rely on individual tasks designed and presented by the researcher. The way in which a student is expected to generalize has already been determined, and the phenomenon being measured is the student's ability to produce a correct and relevant pat-

tern or general statement. If a student is incapable of producing the expected generalization on the given individual task, then the study may report students' failure to generalize.

These methods may fail to capture genuine instances of generalization and transfer not only because the researcher's narrow definition limits the phenomenon in question, but also because generalization and transfer may occur over longer periods of time, in ways that are not easily demonstrated in traditional task protocols:

In the classroom environments in which we now conduct research, an attempt to build theory, a straightforward extension of the laboratory protocol is close to impossible. Such neat slices of life – problem a followed immediately by problem b – are not that often apparent...The understandings that lead to transfer are typically built over extended periods and learners may show evidence of transfer in a variety of ways and in somewhat unpredictable times. This allows a richer picture of emerging competence, but poses methodological problems of no small proportions. (Campioni, Shapiro, & Brown, 1995, p. 39)

The messy nature of learning in classroom settings requires a new approach to studying transfer phenomena. A shift in methodological approach away from the use of discrete tasks may be necessary to capture a wider range of student actions that still legitimately constitute cases of generalization and transfer.

### A Student-Centered Generalization Taxonomy

The taxonomy described in this article extends Lobato's actor-oriented transfer perspective to account for the different types of generalizations students create. Instances of generalization are sought by looking for evidence of students identifying commonality across cases, extending their reasoning beyond the range in which it originated, and deriving broad results from particular cases. Operating from the actor-oriented perspective provides an opportunity to attend to the ways in which learners generate similarities between problems, situations, or contexts. Evidence for generalization is therefore not predetermined, but instead is found by (a) exploring how students extend their reasoning, (b) examining the sense students make about their own general statements, and (c) inquiring into what types of common features students might perceive across cases.

Adopting an actor-oriented perspective requires the researcher to relinquish normative notions of what counts as generalization and instead attempt to take on the student's mathematical perspective. Although mathematical correctness should not be ignored or de-emphasized, limiting the definition of generalization to that of a correct, formal description supports fewer insights into what students themselves construe as general. By better understanding how students generalize their learning experiences, researchers may be able to identify increasing levels of

sophistication in generalizing activity and consequently better support students' development of expertise. Furthermore, capturing how students expand the range of applicability of a general phenomenon can lead to insight about what students have learned about that phenomenon.

The taxonomy that emerged from this study is described in two major results sections. The first section elaborates the system itself, providing descriptions and examples of each of the types of generalizations demonstrated by the study participants. The second section illustrates the power of the taxonomy by elaborating the nature of the evolution of students' abilities to generalize productively. A data episode demonstrating iterative cycles of generalization is included to provide a backdrop against which one can use the taxonomy to examine students' emerging general ideas. The article closes with a discussion of other potential contributions of the taxonomy, both for research on transfer and for generalization in mathematics education.

## METHOD

Two sets of data were collected to gain information regarding the nature of students' generalization in multiple settings. First, a small number of students participated in a teaching experiment to examine their generalizing activity in a nonrestrictive environment. Second, a different group of students participated in a series of interviews to explore their generalizing activity in a typical mathematics class at the same school. Both components of the study were conducted at a public middle school located near a large southwestern city. The school has an ethnically diverse student population—out of its 1,000 students, approximately 40.8% are Latino, 28.2% are White, 16.7% are Filipino, 6.9% are African American, 6.3% are Asian American, 0.7% are Pacific Islander, and 0.4% are Native American. Approximately 15% of the students are English language learners.

### Participants

Seven 7th-grade pre-algebra students were selected to participate in the teaching experiment. The students were recruited, with their teacher's input, on the basis of strong interest in participating in supplemental mathematics lessons, good regular classroom attendance, ability to verbalize their thought processes, and grades of C or higher in their school mathematics classes. A sample of students who displayed medium to high grades, good classroom attendance, and the ability to articulate their thoughts was necessary for the success of the teaching experiment because it was important to include students who were poised to develop new ideas and who could describe their thought processes. Only 7 students from the recruitment pool of 70 volunteered for the course, and every student who volunteered was accepted.



Six students were girls and 1 was a boy. Three students were Latino, 3 were White, and 1 was Asian American. One student was an English language learner and the other 6 were native English speakers. All 7 students were relatively strong proportional reasoners as evidenced by their ability to successfully negotiate a series of proportional reasoning tasks prior to the study. The participants' facility with proportional reasoning likely helped them develop and generalize new ideas about linear functions throughout the course of the teaching experiment.

An additional 7 students were recruited from an eighth-grade algebra class ( $n = 35$ ) to participate in individual interviews. The interview participants were recruited on the basis of the same criteria used for the teaching experiment. The selected students reflected the makeup of the general student population in terms of gender and ethnicity; 3 of the students were boys and 4 were girls. Four of the students were Latino, 2 were White, and 1 was Asian American. Two of the students were English language learners, and 5 were native English speakers. Each student participated in one interview. Gender-preserving pseudonyms have been used for all participants.

### Data Collection and Instruments

All teaching-experiment sessions were taught by the author, who is referred to as the teacher/researcher throughout the rest of the article. All sessions were videotaped and transcribed. The primary purpose of a teaching experiment is for researchers to gain direct experience with students' mathematical reasoning, learning, and development (Cobb & Steffe, 1983). The teaching experiment setting allows researchers to construct models of students' mathematics through the creation and testing of hypotheses in real time while engaging in teaching actions. Through this approach, it is possible to continually develop, test, and refine conjectures about students' generalizations as they solve problems.

The teaching experiment occurred on 15 consecutive school days for 1.5 hr each day. A single camera was placed in the room and run by an observer who was familiar with teaching-experiment methodology in general and the goals of this project in particular. The observer recorded whole-class discussions as well as individual and group work, and he also took detailed field notes. To gain more access into each individual student's understanding, 30-min informal discussions occurred with 1 student at the end of each lesson, resulting in a total of two discussions with each student.

The main purpose of the teaching experiment was to explore the nature and development of students' generalizations as they emerged in the context of realistic problems about linear growth. The learning goals included the development of ratios, the creation of an emergent quantity as the ratio of two initial quantities, and the identification of linear situations as those that have constant ratios. The students worked with gear ratios for the first 7 days of the teaching experiment and a

speed context for the remaining 8 days. Two physical artifacts ultimately proved important in influencing how the students reasoned. The first was a set of physical gears that the students could directly manipulate to experiment with ways of coordinating rotations. The second artifact was a computer program called SimCalc Mathworlds (Roschelle & Kaput, 1996), which simulated speed scenarios showing two characters walking across the screen at constant speeds. The use of the software allowed the students to create and test conjectures about how changing distance and time would affect the character's speed.

Figure 1 provides an overview of the activities students engaged in, and the mathematical ideas addressed as they were explored in the classroom during the observed unit.

The individual interviews lasted 60 min and were videotaped and transcribed. The goal of the semistructured interviews (Bernard, 1988) was to determine what sense the students made of the generalizations they developed, what types of explanations students provided for them, and what types of extensions and limitations students saw for their own generalizations. Thus, the model for the interviews involved taking some of the general statements students had developed in class and devising task questions addressing them. Transcription of the interviews as well as the teaching-experiment sessions captured not only the students' utterances, but also their written explanations, drawings, and gestures. Descriptions of students' gestures or drawings as they interacted with physical artifacts and representations served as an additional source of information about students' reasoning as they generalized.

The interview sessions occurred during a 10-day unit on linear functions, in which the classroom teacher relied on the use of several activities from the Connected Mathematics Project (Lappan, Fey, Fitzgerald, Friel, & Phillips, 1998). Connected Mathematics Project is a problem-centered curriculum in which mathematical ideas are embedded in realistic and engaging problems designed to help students develop both understanding and skill. The official text for the course was a more traditional Algebra I book (Larson, Boswell, Kanold, & Stiff, 2001), on which the classroom teacher occasionally relied for practice problems and homework. Based on the student's work, the tasks and the interviewer's questions were varied, but each task required the student to extend his or her reasoning to a larger set of cases or numbers. Figure 2 includes sample interview items posed to students.

## Data Analysis

Analysis of the data followed the interpretive techniques of grounded theory, in which categories of generalizations were induced from the data (Glaser & Strauss, 1967; Strauss & Corbin, 1990). Because the teaching experiment occurred 4 months prior to the individual interviews, transcripts of the teaching-experiment

Day	Situation	Mathematical Topics	Class Activities
1	Gear Ratios	Coordinating quantities	Finding ways to keep track of simultaneous rotations of different-sized gears
2		Relating teeth to rotations; Inverse relationships	Determining how to relate the turns of a gear with 8 teeth to a gear with 12 teeth
3		Constructing ratios; Constant ratios in non-uniform tables	Finding relationships between 8/12/16 gears; determining if rotation pairs come from the same gear pair
4		Connecting $y = ax$ equations to the gear situation	Explaining how $(3/4)m = b$ relates to both rotations and teeth
5		$y = ax + b$ gear situations	Modeling situations in which A turns before connecting to B
6		Representing $y = ax + b$ situations in tables	Making $y = ax + b$ tables; comparing and contrasting to $y = ax$ tables
7	Gear Ratios / Speed	Non-uniform $y = ax + b$ tables; Isolating quantities for speed	Determining constant ratio from $y = ax + b$ tables; Who walks faster, Clown or Frog
8	Speed	Changing initial quantities without changing the emergent quantity	Finding as many ways as possible to make Frog walk the same speed as Clown
9		Classes of equivalent ratios	Explaining why equivalent ratios mean the same speed
10		Constant ratios in non-uniform tables	Determining if Frog went the same speed given values in tables
11		Connecting $y = ax$ equations to the speed situation	Explaining how $(2/3)c = s$ represents speed
12		$y = ax + b$ speed situations and tables	Modeling situations in which Clown starts away from home and walk constant speed; Making tables to represent $y = ax + b$
13		Non-uniform $y = ax + b$ tables	Deciding constant speed from $y = ax + b$ tables
14		Non-uniform $y = ax + b$ tables	Deciding constant speed from tables; describing $y = ax + b$ speed situations
15	Student-Invented	Meaning of Linearity	Inventing situations involving linear relationships

FIGURE 1 Overview of the teaching experiment unit.

**Task 1 (Linear Patterns in Tabular Form Developed in Class)**

In class you made a table for the equation  $y = 600 - 500x$ . Can you recall the table?

$x$	$y$
0	600
1/4	475
1/2	350
3/4	225
1	100

A. What sorts of patterns did you notice in class, and do you notice now?

- Why does  $y$  decrease by 125 each time?
- Does it matter what the  $x$ -values are?
- Is there a relationship between the 1/4 and the 475?
- Does your pattern always hold?

B. Do you think this pattern could continue?

C. Can you make additional entries in the table?

**Task 2 (Real-World Pattern of Linearity: Bridges and Pennies)**

In class you made paper bridges to see how many pennies they would hold. (Say students had decided that adding an extra layer would result in the bridge holding 10 more pennies).



A. What determines how much weight the bridge can hold?

B. Does it matter how long the bridge is? Would a long bridge hold more pennies, or would a short bridge hold more, or does it not matter?

C. What is the relationship between the layers of paper and the pennies on the bridge?

D. Why does adding an extra layer to the bridge result in holding 10 more pennies? (Could it just as easily be 9 or 11? Will it always have to be exactly 10?)

E. Will this relationship always hold, or does it matter what numbers you use? (Ask about 20 layers, 10,000 layers, and part of a sheet of paper instead of a whole layer).

F. How many pennies would a bridge with 25 layers be able to hold?

G. Say you had 115 pennies that you wanted to put on a bridge. How many layers would your bridge need?

FIGURE 2 Sample interview items.

data were coded first. The initial coding pass relied on open coding, in which instances of generalization were identified as they fit the definition described earlier (the identification of commonality across cases, the extension of reasoning beyond the range in which it originated, & the derivation of broad results from particular cases). Classroom sessions data were first analyzed chronologically day-by-day, for the purpose of finding evidence to determine (a) the apparent meaning of the

identified generalization to the student, (b) the basis for each generalization, and (c) the range of extension of each generalization. Through this process, emergent categories of generalizations were developed.

Data from the teaching experiment were then re-analyzed student by student. Sources of the different types of generalizations were sought by identifying the actions in which the students had engaged to formulate each generalization. Categories were further tested, and appropriate modifications were made through coding the individual interview data. When categories of types of generalizations developed from the teaching-experiment data did not fit with individual interview data, new categories emerged. These new categories were then subjected to subsequent passes through both the teaching-experiment data set and the interview data set until theoretical saturation had been achieved. All generalizations detected in both the teaching experiment and the individual interviews were ultimately recoded with the final categorization scheme.

Each category emerged from analyzing multiple pieces of evidence across groups, students, and tasks. The following data excerpts presented, however, are those that provide the clearest examples of the phenomena in question. For brevity's sake, only one or two excerpts are included, which do not encompass the range of data analyzed to develop the category.

## RESULTS

The goal of the study was to develop a coherent, empirically grounded taxonomy capturing a broad range of generalizing actions and reflection generalizations. Although the taxonomy is the result of students working with many different problem tasks, it is not a categorization of task types. Instead, it captures students' generalizing acts as they reason with a range of problems. The results are reported in two major sections. The first section introduces the Generalization Taxonomy, providing definitions, descriptions, and examples of each category of generalization demonstrated by the study participants. The second section presents a data episode that demonstrates iterative cycles of generalizing as viewed through the lens of the taxonomy.

### Part 1: The Two-Part Generalization Taxonomy

The more than 300 generalizations coded fell into one of two major categories, generalizing actions and reflection generalizations. Generalizing actions describe learners' mental acts as inferred through the person's activity and talk. An examination of problem-solving behavior, such as the mathematical operations a student employs while working with a problem, a student's apparent mathematical focus, the properties and relations a student attends to, or the strategies in which a student engages, can lead to a description of the types of mental actions the student appears

to employ in his or her attempts to generalize. The behaviors themselves do not constitute the generalizing actions, but contribute to the researcher's determination of what type of generalizing action the student may be performing.

The notion of a mental action is a researcher's construct. From a Piagetian perspective, knowledge arises from a learner's activity, either physical or mental; it is goal-directed activity that gives knowledge its organization (von Glasersfeld, 1995). Mental action is distinguished from physical action, but the term *action* is also deliberate – it emphasizes the belief that learners are nonpassive, adaptive beings who construct knowledge through interaction with their experiential worlds. The inferred mental acts are characterized as generalizing actions in part to distinguish them from the verbal or written expressions that comprise students' reflection generalizations. However, it will not be possible to identify with certainty every mental process in which a student engages. The notion of generalizing action is only one way of making sense of an underlying process that could result in particular behaviors.

Students' public statements were categorized as reflection generalizations. If a student explicitly stated a common property, pattern, or relation of similarity, the statement was recorded as a reflection generalization. Instances in which a student did not produce a verbal or written statement but used the result of a generalization, such as applying an idea from a prior situation to a new problem, were also categorized as reflection generalizations. In both cases, behaviors or statements coded as reflection generalizations represent the student's ability to either identify or use a generalization he or she has created.

When students made a statement of generalization, it was often possible to trace their reasoning back to the point at which they began to engage in the types of generalizing actions that led to their final statement. By attending to the difference between students' generalizing actions and reflection generalizations, it was possible to identify which actions were most often connected to particular types of general statements. Categorizing actions and reflections separately also allowed for the identification of iterative "action → reflection → action → reflection" cycles of reasoning, in which students' generalizations evolved in sophistication over time.

The two major categories of generalizing actions and reflection generalizations are described in turn. Each category contains multiple subcategories, in which different types of generalizations are presented and explained through data excerpts from both the teaching experiment and the individual interviews.

### *Generalizing Actions*

Students' generalizing actions fell into three major categories: relating, searching, and extending. Figure 3 provides an overview of the three categories.

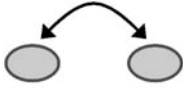
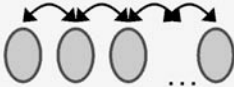

GENERALIZING ACTIONS		
<p><b>TYPE I: RELATING</b></p> 	<p>1. <i>Relating Situations</i>: The formation of an association between two or more problems or situations.</p>	<p><i>Connecting Back</i>: The formation of a connection between a current situation and a previously-encountered situation.</p>
		<p><i>Creating New</i>: The invention of a new situation viewed as similar to an existing situation.</p>
	<p>2. <i>Relating Objects</i>: The formation of an association of similarity between two or more present objects.</p>	<p><i>Property</i>: The association of objects by focusing on a property similar to both.</p>
		<p><i>Form</i>: The association of objects by focusing on their similar form.</p>
<p><b>TYPE II: SEARCHING</b></p> 	<p>1. <i>Searching for the Same Relationship</i>: The performance of a repeated action in order to detect a stable relationship between two or more objects.</p>	
	<p>2. <i>Searching for the Same Procedure</i>: The repeated performance of a procedure in order to test whether it remains valid for all cases.</p>	
	<p>3. <i>Searching for the Same Pattern</i>: The repeated action to check whether a detected pattern remains stable across all cases.</p>	
	<p>4. <i>Searching for the Same Solution or Result</i>: The performance of a repeated action in order to determine if the outcome of the action is identical every time.</p>	
<p><b>TYPE III: EXTENDING</b></p> 	<p>1. <i>Expanding the Range of Applicability</i>: The application of a phenomenon to a larger range of cases than that from which it originated.</p>	
	<p>2. <i>Removing Particulars</i>: The removal of some contextual details in order to develop a global case.</p>	
	<p>3. <i>Operating</i>: The act of operating upon an object in order to generate new cases.</p>	
	<p>4. <i>Continuing</i>: The act of repeated an existing pattern in order to generate new cases.</p>	

FIGURE 3 Generalizing actions.

By distinguishing between these three actions, the taxonomy accounts for differences in mental activities as students generalize. The three categories are not mutually exclusive; students' actions were instead classified as falling into a particular category based on their primary focus when reasoning. Each category and its subcategories are defined and described in more detail following.

*Type 1: Relating*

Relating occurs when a student creates a relation or makes a connection between two (or more) situations, problems, ideas, or objects. A student may see a situation and make a connection to a prior situation seen before, or may see a situa-

tion and then generate another situation that he or she views as similar to the first. In either case, when a student engages in relating, he or she perceives a relation of similarity between two or more situations, but might not necessarily be able to elaborate how the situations are connected. Students who engaged in relating made connections between situations and connections between objects.

*Relating situations.* The act of relating situations involves the formation of an association between two or more problems or situations. According to the actor-oriented perspective, what constitutes a “situation” is dependent on the student’s perception; it can refer to the context surrounding a problem, the combination of circumstances at a given moment, or the setting in which one engages in a particular type of reasoning. Therefore, if a student perceives two situations to be distinct, and then establishes a relation of similarity between them, he or she is relating situations even if the contexts are identical from a researcher’s perspective. For example, students in the teaching experiment worked with the following problem: “Gear A has 5 teeth and Gear B has 8 teeth. Gear A starts spinning on its own, before it’s connected to Gear B. It spins 6 times. Then Gear B is plopped down and they spin together.” Larissa remarked:

Larissa: Oh! That’s like, that’s like the test, the swimming laps lady!

Timothy: Yeah, yeah!

Larissa: Then he jumps in 5 minutes later and ...

Timothy: We did this test where, oh this person jumps in and does a certain amount of laps and then 5 minutes later...and then doing a rate of this, and 5 minutes later, this guy jumps in doing this rate.

Larissa and Timothy seemed to view the gear problem, in which one gear began to spin before the other joined it, as similar to a problem they had seen before in which one person began to swim at a particular rate, and was later joined by another swimmer, swimming at a different rate. Because the students made a connection to a previously encountered problem, their actions are coded as connecting back, even though there are important mathematical differences between the two scenarios. Students may also connect back when noticing a property in a current situation that reminds them of a similar property from a prior situation, or when focusing on a feature of a problem that they perceive as similar to a feature from another problem.

Dani, a teaching-experiment student, also generalized by relating situations, but did so in a different manner. She worked with the following table (Figure 4) to determine if data would be linear or nonlinear. The table showed how far a clown walked during the associated amount of time for different portions of his journey:

Dani decided that the clown was not walking the same speed, and the teacher/researcher (TR) asked her to explain her thinking:



Distance	Time
1 cm	12 sec
2 cm	6 sec
3 cm	4 sec
4 cm	3 sec
6 cm	2 sec
12 cm	1 sec

FIGURE 4 Nonlinear table of Clown's times and distances.

Dani: The more seconds he has, he slows down. Let's see. The more seconds he has, he'll slow down. And the less seconds he has, he'll speed up faster.

TR: Does that mean that it's linear or nonlinear?

Dani: It really means it wasn't linear.

TR: Okay, and how come?

Dani: You know how, like if you had less time to go into the grocery store to get the foods on the grocery list, you would go faster if you had like 1 s to do it in. You would like be in and out real quick. Same thing here. 12 s you would take, like, not as much time to get all the groceries. You would like slow down your walk a little to get there.

In an attempt to explain why she thought the data were not linear, Dani invented what she viewed as an analogous situation in which one would move through a grocery store either quickly or slowly depending on the amount of time available. Given Dani's description, she appeared to take two pairs from the table, (12 cm, 1 s) and (1 cm, 12 s) and then describe each through the grocery-store scenario. Because Dani only mentioned time, it is difficult to tell whether she held distance constant or also varied distance. Although this may not constitute a valid analogy to a researcher, Dani apparently perceived a similarity between the table and her invented scenario, and thus her actions are coded as creating new rather than connecting back. Unlike the students' actions in the gears and swimming example, Dani deliberately invented a new situation to achieve the specific goal of justifying her conclusion that the data were not linear.

*Relating objects.* Two teaching-experiment students, Larissa and Julie, shared equations they created to describe the journey of a character who walked at a constant speed after starting 6 cm away from home. Larissa wrote " $10[(c - 6) \div 12] = s$ " and Julie wrote " $\left[ \frac{(\# \text{ of cm away from home} - 6)}{12} \right] 10 = \text{how long it took him.}$ " The students remarked that the two equations shared a connection:

Larissa: They're both generalizing.

Timothy: They're both the same thing though. They're both similar, but just done in a different form.

TR: What did you mean when you said they're both generalizing?

Larissa: They're both telling you how you can solve the equation even though they're different.

In their attempt to describe the property embodied by both sets of equations, the students focused on a similar property, namely, that both equations determined missing values. These students' actions are coded as relating objects, because they formed an association between two or more present mathematical objects, such as equations, graphs, tables, or other representations. In contrast to the prior category of relating situations, students relating objects do not consider the two objects in question to be associated with different contexts or situations. One might notice that an equation has the same structure as a different equation, or that two graphs share a similar feature. Timothy and Larissa related objects by focusing on a property similar to both equations.

One may also relate objects by focusing on the syntactic form taken by two or more expressions. For example, Larissa wrote  $\frac{1}{4}C = \frac{3}{5}S$  on the board whereas

Dora wrote  $S = \frac{5}{12}C$ . Larissa then remarked that the equations shared a similarity: "Those are both divided." She focused on the form of both equations, noticing that both contained division, and related the two as similar based on their form.

### *Type 2: Searching*

When searching, a student will perform the same repeated action in an attempt to determine if an element of similarity will emerge. Teaching-experiment students worked with the following table (see Figure 5) to see if the clown walked at a constant speed or a variable speed:

Timothy divided centimeters by seconds for each pair. He took 7 and 1/2 and divided it by 5, took 27 and divided it by 18, and repeated this same action for each pair in the table. He concluded "They all have the same relationship between the seconds and the centimeters." When asked what the relationship was, he said "ei-

<i>CM</i>	<i>Sec</i>
7 1/2	5
27	18
4 1/2	3
16	10 2/3
1/10	1/15

FIGURE 5 Linear table of Clown's times and distances.

ther if you look at it centimeters to seconds, the centimeters are 1 and 1/2 of the seconds, or the seconds are 2/3 of the centimeters.”

In contrast to the students’ relating actions discussed previously, Timothy’s searching actions were deliberate and goal-oriented. He focused on a mathematical relationship between two objects, and thus his actions were coded as searching for the same relationship. When searching, a student will perform the same action, such as calculating a ratio, on (usually) multiple pairs of numbers. Through this repeated action, the student seeks to find an element of similarity. The student’s goal is therefore to locate this element of sameness. He or she may not know if any relation of similarity exists, or what it might be, but by searching the student anticipates that he or she may find one. Thus with searching actions, students do not aim to create a relation of similarity so much as they aim to find what they anticipate as a preexisting element of sameness across multiple instances.

As shown in the aforementioned excerpt, searching actions occurred most often when students worked with tables containing many pairs of numbers. They would frequently look for an element of sameness across all of the pairs in a given table. A student’s focus on sameness can include anything that he or she views as the same, even if it may be mathematically trivial from the researcher’s perspective. In addition to focusing on relationships, students also focused on procedures, patterns, and solutions when searching.

*Same procedure.* As Dani worked with a table (see Figure 6) representing the clown’s total distance from home and his time walking, she engaged in actions that, on the surface, appeared similar to Timothy’s actions:

Dani: Oh yeah, I was finding the gaps between all the numbers, like negative 5 and 20 and 2 and 12 and so on. For both sides. And when I got both I divided 25 by 10 and 20 by 8 and 12.5 by 5 and 87.5 by 35 and it all gave me 2.5.

Although Dani’s actions were computationally sophisticated, the procedure she described appeared to lack a connection to any speed relation. Specifically, Dani was not able to indicate what the differences that she had calculated represented, and she could only justify her procedure by explaining that it must be right “be-

total cm	total (Sec)
25 - 5	27 10
20 20	12 8
20 10	20 8
12.5 52.5	25 5
87.5 190	60 35

FIGURE 6 Dani’s table of Clown’s total distances and times.

cause that's what Larissa did yesterday." Dani's actions were therefore coded as searching for the same procedure rather than searching for the same relationship.

It can be difficult for the researcher to distinguish between the search for a relationship versus a procedure. One could argue that Timothy performed a procedure by dividing the centimeters by the seconds throughout the table. However, Timothy's arithmetic calculations appeared to be tied to his quantitative comprehension of the situation (Thompson, 1994), as indicated in part by Timothy's subsequent discussion of the centimeters–seconds relationship in terms of the character's speed. In contrast, Dani's actions did not appear to be connected to reasoning about a relationship between quantities. Thus, the distinction between the search for the same relationship versus the search for the same procedure rests with the researcher's judgment of the student's understanding of the subject matter.

*Same pattern.* When a student focuses on the same pattern, he or she might notice a pattern and then search to determine if it remains stable. For example, Mario, an interview participant, focused on a pattern in a table introduced by his classroom teacher in class (see Figure 7):

Tracing his finger down the  $y$ -column, Mario noticed regularity in the differences between successive values: "on the  $x$  side it's going up by ones and on the other side it's going up by...sevens." Mario's action of attending to the differences down the column revealed a stable pattern in the data. However, his language did not indicate attention to the relation between the increase in the  $x$ -values and the corresponding increase in the  $y$ -values. Because Mario did not attend to how the values in the table were coordinated, the simple numerical pattern he identified did not represent a search for the same relation. His focus was on the pattern only.

*Same solution or result.* The interview participants worked with the equation  $2(4b - 3) = 8b - 6$  with their classroom teacher:

$x$	$y$
0	5
1	12
2	19
3	26
5	40
6	47

FIGURE 7 Table introduced in the classroom.

Mrs. D: I heard something over here about  $b$  equals  $b$ . And that may have raised the question. You find that it is one other solution. Pick another number. This side of the room, you pick a positive value between 1 and 5. Okay, you folks (gesturing to the other side) pick a negative value between negative 1 and negative 5. And you folks (gesturing to the students in the middle) positive value between 1 and 5. Substitute it into the sentence. Okay? If the value you choose is a solution, what will be true?

Students: Equal.

Mrs. D: What will be true of both sides of the equation? They'll be what? The same, they'll be equal. That's what an equation is all about.

The classroom teacher then asked various students in the class to share their work given that they had picked arbitrary numbers from negative 5 to 5 to substitute for  $b$ . Each student reported that his or her value resulted in a true statement. The students noticed that in every case, the statement was true. One student, Aditi, remarked to her neighbor, "I think every number is going to work."

Mrs. D: Okay, what does that hint to you? How many solutions might there be?

Students: Infinity.

Mrs. D: Negative 6 was a solution, but there might be infinitely many others.

As a group, the students engaged in a repeated action by substituting various arbitrary values into the equation. Each time they found the same result, and with their classroom teacher's prompting, they generalized that any value could result in a true statement. The students' focus remained on the outcome of their actions, and they made inferences based on determining the same outcome repeatedly. When students engage in searching for sameness via the same procedure, relationship, or pattern, their focus centers on their own repeated actions more than on the outcome of those actions. In contrast, when students search for the same solution, they focus on the result of their actions.

### *Type 3: Extending*

Generalizing activity falls into the extending category if a student not only notices a pattern or a relationship of similarity, but then expands that pattern or relationship into a more general structure. When extending, a student expands his or her reasoning so that it reaches beyond the problem, situation, or case in which it originated. Through this action the student generates something new, such as a new domain of validity, new members of a class, a new relationship, a new structure, or a new description of a general phenomenon.

For example, Timothy graphed a linear speed situation in the first quadrant and shared his results with fellow students. Larissa, on observing the graph, remarked

“Going back to the how many points can you...in fact, it could also go to a four quad graph.” Her comment referred to the idea that Timothy did not need to restrict his line to the first quadrant, but could extend it beyond the few points he had originally graphed. In noting that the graphical representation could be applied to a larger range of cases shown, Larissa extended her reasoning by expanding the range of applicability.

Notice that in the relating category, students sometimes create new situations. If a student focuses on the generality of an idea in a way that extends beyond any one or few particular instances, problems, or situations, then his or her actions would be categorized as extending. If, however, a student’s attention is focused primarily on the formation of an association between two situations, then his or her generalizing action is categorized as relating. This is an example of the nonexclusivity of the three categories; student’s actions were categorized based on the main focus of their reasoning. Data from the study revealed that in addition to expanding the range of applicability, students also extended by removing particulars, operating, and continuing.

*Removing particulars.* Students who extend by removing particulars take what they have generalized within a particular problem and then remove some contextual details from their description of a property, relationship, pattern, or other phenomenon to state a global case. One could also go on to then describe a class of particular objects and a general phenomenon that would be true for each object in the class. For example, students in the teaching experiment worked with a situation in which a clown walked 12 cm in 10 s. Dora wrote a table on the board in which the centimeters increased by 1 and the seconds increased by  $5/6$ . Julie then wrote a table in which the seconds increased by 1 and the centimeters increased by  $6/5$ . Timothy remarked:

Timothy: If you’re going up by 1 in seconds, it’s going up by  $6/5$ . If you’re going up by 1 in centimeters, it’s going up by  $5/6$ , or the inverse.

Timothy then thought for a time about this idea and extended his reasoning beyond the particular speed situation:

Timothy: Whatever 1 you go up by, and you go up by something else on the other one, if you were to switch it around and go up by 1 on the other one, you would go up by the inverse.

TR: For any table?

Timothy: I’m guessing it may. I’m not sure.

Timothy’s thought process was categorized as extending because he reasoned about how linear tables could be organized in a way that was no longer tied to the particular example in which the clown’s speed was  $6/5$  cm per second.

*Operating.* A student extends by operating when he or she performs an operation on a relationship or a pattern to develop new instances of a phenomenon. Students' actions were only categorized as extending by operating if the operation in question changed the relation, pattern, or representation. For example, one could take a ratio and multiply it by any real number to generate many instances of equivalent ratios. By multiplying a given ratio  $a:b$  by a constant  $c$ , one has altered the representation of the existing ratio and expressed it as  $ca:cb$ . One could also engage in a subset of this action, such as doubling or halving a composite unit (Lamon, 1995). When extending by operating, a student deliberately applies a mathematical operation to an object to generate new cases.

Manuel, an interview participant, extended by operating when he worked with the following table (see Figure 8) representing the relationship between the layers of paper constituting a bridge and the number of weights a bridge could hold:

When asked if fractions could work in the table, Manuel said yes, it would be possible to write 8 and  $1/2$ . The interviewer then asked:

Int: So what if we had 8 and  $1/2$  for  $x$ ? Would it be possible to figure out what  $y$  was?

Manuel: Hmm. 64. I think it's 64.

Int: Why is that?

Manuel: Because I figured that half of 7, it's 3 and  $1/2$ . So when I added the 7 I added an extra, an extra 3.5.

Int: So you added 3.5 to what?

Manuel: Oh hold on. 64 and  $1/2$ .

Int: So why does that work?

Manuel: Because when you, because this is a half right there, so then I took a half from 7, and when I, when I added the 7, I added an extra 3.

X	Y
0	5
1	12
2	19
3	26
4	33
5	40
6	47
7	54
8	61

FIGURE 8 Manuel's table comparing layers and weights.

Manuel formed a 1:7 unit by coordinating the differences between  $x$  and  $y$  values in the table, operated on the 1:7 unit by halving 1 and halving 7, then adding .5 and 3.5 to get 8.5 and 64.5. He has formed a broader set of equivalent ratios by operating on the 1:7 unit.

*Continuing.* Another interview participant, Mario, detected a pattern in Figure 9:

Mario: Um, well, as  $x$  goes up by 1, it's...hmm. I don't know, it goes, see it goes 1, 2, so there's 1 in between that, and then there's 2, and then there's 3, and then there's 4 so they're, the intervals are the, the same, like it... Well they're, they're not the same but there's like a pattern of how they go up.

Int: Hmm.

Mario: It's not the same number in between them but, um, the pattern like is the same going up...if you did 6 it'd go to 16, so it's plus 5. Then you go to 7 it'd be, it'd be like 22. So, um, there is a pattern on how  $y$  goes up and how  $x$  goes up.

Mario continued the pattern of increases of 1 in each interval, and was able to extend this to generate the new pairs (6, 16) and (7, 22). He focused on the pattern in the table rather than on a relationship between  $x$  and  $y$ , which he could not discern. Unlike Manuel, Mario did not operate on the pattern as he continued it. Students extend by continuing when they repeat a pattern without changing it. A student's focus when extending by continuing is slightly different from his or her focus when extending by operating. When operating, a student's focus rests with the ratio or other relationship, such as Manuel's focus on the 1:7 unit shown previously. When continuing, a student's focus rests with the pattern itself rather than the relation that causes the pattern.

### *Reflection Generalizations*

Recall that reflection generalizations are students' final statements of generalization; they represent either a verbal statement, a written statement, or the use of

X	Y
1	1
2	2
3	4
4	7
5	11

FIGURE 9 Nonlinear table given to Mario.



the result of a generalization. Reflection generalizations are thus tightly linked with students' generalizing actions. Figure 10 shows the three major categories of reflection generalizations that emerged from the data analysis.

The data excerpts will demonstrate that many reflection generalizations mirror generalizing actions. For example, statements of sameness often accompany the generalizing action of relating, and statements of general principles often accompany the generalizing action of searching. The actions of noticing similarity, generating analogous situations, or searching for similarity frequently result in declarations of sameness or articulations of rules and principles. Each of the three categories, Identification or Statement, Definition, and Influence, is described in detail in the sections following.

### Identifications or Statements

When a student identifies a generalization, he or she may refer to a general pattern, property, rule, or strategy, or he or she may explicitly identify a common element across different cases or problems. Students who produce identifications or statements make their generalizations public, either verbally or in written form, as

REFLECTION GENERALIZATIONS	
<b>TYPE IV: IDENTIFICATION OR STATEMENT</b>	1. <i>Continuing Phenomenon</i> : The identification of a dynamic property extending beyond a specific instance.
	2. <i>Sameness</i> : Statement of commonality or similarity.
	<i>Common Property</i> : The identification of the property common to objects or situations.
	<i>Objects or Representations</i> : The identification of objects as similar or identical.
<b>TYPE V: DEFINITION</b>	<i>Situations</i> : The identification of situations as similar or identical.
	3. <i>General Principle</i> : A statement of a general phenomenon.
	<i>Rule</i> : The description of a general formula or fact.
	<i>Pattern</i> : The identification of a general pattern.
<b>TYPE VI: INFLUENCE</b>	<i>Strategy or Procedure</i> : The description of a method extending beyond a specific case.
	<i>Global Rule</i> : The statement of the meaning of an object or idea.
	1. <i>Prior Idea or Strategy</i> : The implementation of a previously-developed generalization.
	2. <i>Modified Idea or Strategy</i> : The adaptation of an existing generalization to apply to a new problem or situation.

FIGURE 10 Reflection generalizations.

a description or a mathematical statement. One may identify a continuing phenomenon, an element of sameness, or a general principle.

*Continuing phenomenon.* When students made statements of continuing phenomena, they typically identified properties that extended beyond a specific instance. Such statements were characterized by a sense of continuation, motion, or dynamic relation between quantities. Study participants produced statements such as “it keeps going,” “it always happens no matter what,” or “for every  $a$ , you see  $b$ .” For example, Larissa worked with the table shown in Figure 11 to determine if a character walked a constant speed:

After having engaged in the generalizing action of searching for the same relation, Larissa concluded, “Every time it goes up 1, it goes up by 1 and  $3/5$ .” Her attention remained focused on the stable pattern that continued across a series of regular iterations.

*Sameness.* Students who produced statements about sameness explicitly identified a property that they perceived as linking two or more problems, situations, or objects. At times, students focused on the property that they perceived as similar. Other times, students focused on the actual objects, representations, problems, or situations that they perceived as the same. Based on the student’s primary focus as evidenced by his or her description, reflection generalizations were categorized either as (a) identification of a common property, (b) identification of same objects or representations, or (c) identification of same situations.

For example, Carla, an interview participant, worked with the following table (Figure 12) to determine if there was a relation between  $x$  and  $y$ :

Carla engaged in the generalizing action of searching for the same relationship by dividing  $y$  by  $x$  for each ordered pair. She was then able to state what each pair held the same: “Each one,  $x$ , is a third of  $y$ .” Her identification of the common property across all pairs was the reflection generalization that followed her generalizing action of searching for the same relation.

In contrast, recall the example in which Larissa and Julie wrote their respective equations on the board: “ $10[(c-6) \div 12] = s$ ” and “ $\left[ \frac{(\# \text{ of cm away from home} - 6)}{12} \right]$  10 = how long it took him”. Timothy’s remark that “They’re both the same thing

Total CM	Sec
13.25	10
17	16
22.625	25
32	40
44 $1/2$	60

FIGURE 11 Table showing  $y = mx + b$  relation.

$x$	$y$
3	9
-1	-3
$60$	180
4	12

FIGURE 12 Table of  $x$ - and  $y$ -values.

though. They're both similar, but just done in a different form" represents the identification of same objects or representations. He engaged in the generalizing action of relating objects to reach his conclusion; Timothy's actual statement was the reflection generalization.

An identification of same situations occurred in the first example in this article in which Timothy and Larissa recalled that the gear problem was like the swimming problem. When Larissa noted, "that's like the test, the swimming laps lady" she identified the swimming situation as similar to the gears situation. Her generalizing action that prompted the statement was relating (connecting back), and the result of her action was the reflection generalization, the statement in which she explicitly identified the two problems as similar.

*General principles.* Study participants identified general rules, patterns, strategies, and global rules. When these statements occurred in algebraic form, they represented the type of generalization that many researchers seek and recognize as valid. Students in the study, however, produced reflection generalizations about general principles in several different forms. These statements fell into four major categories: (a) general rules, (b) patterns, (c) strategies or procedures, and (d) global rules.

Statements of general rules occurred as verbal descriptions, written expressions, and algebraic descriptions of rules that described mathematical relationships. For example, when working with a gear problem, Ming described a rule verbally: "You times the number of small gear turns by two thirds and you would get the number of the big gear." Larissa later expressed the same idea algebraically with the equation  $s \cdot (\frac{2}{3}) = b$ . Both students engaged in the generalizing action of searching for the same relationship before eventually reaching the point at which they could explicitly state the rule.

Students who identified general patterns described a mathematical pattern either verbally or in written form. They often produced these statements after engag-

ing in the generalizing action of searching for the same pattern. Recall the prior example in which Mario focused on a pattern in a well-ordered table with differences of 7 between successive  $y$ -values (see Figure 7). He traced his finger down the  $y$ -column and identified the pattern “on the  $x$  side it’s going up by ones and on the other side it’s going up by...sevens.”

Students who stated general strategies or procedures were able to describe their strategies in ways that were no longer limited to actions performed in one specific case. Instead, the description could pertain to a general class of problems. For example, Timothy was asked how he could determine, given a table of distances and times, if a character walked the same speed or not. Timothy described what he did with each table, explaining “You can find the smallest whole-number pair, and like either put one over the other and simplify and see if all of them do the same thing.” Timothy’s method involved calculating a ratio based on the smallest ordered pair, and then taking the ratio of every other pair to see if they were equivalent. Timothy’s description of his strategy was not limited to one particular table or speed, but instead represented a statement of a more general approach.

Statements of global rules are generalizations about the meaning of mathematical objects or ideas. Global rules are not limited to specific cases or types of situations; instead, they represent a student’s more general understanding of an idea, such as what constitutes linearity or what slope represents. For example, Juanita, an interview participant, developed a rule that a table of data must contain three patterns for the data to be linear:

Juanita: For it to make a straight line there has to be a pattern this way (gesturing down the  $x$  column), a pattern this way (gesturing down the  $y$  column), and a pattern going back and forth right here (gesturing across the  $x$  &  $y$  columns).

Even though this rule is only valid for well-ordered tables, it was general for Juanita and she used it to make decisions about whether data in a table would result in a straight line. Because Juanita viewed it as a valid and general rule, it was categorized as a reflection generalization despite its limited correctness.

### *Definitions*

Cases in which students made statements conveying the fundamental character of a pattern, relation, class, or other phenomenon were characterized as definitions. For example, Dora worked with two gears, one with 8 teeth and one with 12 teeth. She had already generalized that regardless of the particular number of rotations the connected gears would make, the larger gear would rotate two thirds as many times as the small gear. When asked to think about whether there could be any other sized gears that could demonstrate the same ratio, Dora eventually realized that there were multiple cases of gears that would fulfill this relation:

Dora: Probably like if you, so it was two thirds. You can get anything that is equal to two thirds.

Dora referred to the idea that any two gears with a 2:3 ratio of teeth would also have a 2:3 ratio of revolutions. Thus she defined a class of gear pairs that would satisfy the two thirds rotation relation.

### *Influence*

There were cases in which students implemented previously developed generalizations in new problems or contexts. The influence of a previously developed generalization on new activity was also categorized as a reflection generalization, adhering to the requirement that reflection generalizations represent one's ability to either identify or use his or her created generalization. A student may implement a prior idea or strategy, or may modify a prior idea as he or she approaches a new problem.

*Prior idea or strategy.* The influence of prior activity can be very difficult to categorize unless the researcher is familiar with the student's history and the history of the student's prior instruction. To categorize these instances, cases were sought in which it was possible to detect the trace of a student's prior experience on his or her current behavior. In one example, students worked with a table of clown's total distances from home and times (Figure 13). Julie explained that she initially thought the clown's speed would be 4/9: "At first I thought it was 4/9, but I tried my equation and it didn't work."

Julie's first reaction to the table was to focus on the (9,4) pair and conclude that it represented the clown's speed. Julie's attention to this pair suggests the influence of the prior gear problems. When the students worked with the gears and created tables of rotations, they fixated on the smallest whole-number pair in the table. The students eventually generalized that the smallest whole-number pair represented the gear ratio. Their attention to this pair in every gear rotation table, in combination with other studies showing that students reasoning in speed situations do not behave this way (Lobato & Siebert, 2002; Lobato & Thanheiser, 2000, 2002), suggests the influence of prior activity on the new problem. Although Julie's initial at-

Total cm	Time (sec)
4	0
9	4
16.5	10
34	24
41 1/2	30

FIGURE 13 Total distances and times for Clown.

tempt was unsuccessful, her attention to  $4/9$  was still categorized as a generalization because she approached the distance-time table differently thanks to her experience with gears.

*Modified idea or strategy.* Once Julie and Larissa realized that  $4/9$  was not the clown's speed, they adapted their first idea. Larissa explained: "And then I realized that you had to subtract four from nine to get five and four, and that was, and then that switched is  $4/5$ ." Larissa subtracted four because the clown started 4 cm away from home, and then tried  $4/5$  sec per cm as his speed. She compensated for the extra +4, adapting the prior strategy developed in the gear situation to account for the extra constant. The reflection generalization is Larissa's adapted strategy.

### *Relationships Between Generalizing Actions and Reflection Generalizations*

At times students' reflection generalizations simply represented the public utterance of the generalizing action in which they had engaged. For instance, Carla's identification of the common property that "Each one,  $x$ , is a third of  $y$ " was a verbal description of what she had found by searching for the same relation. Other times, a reflection generalization represents a step beyond the generalizing action by formalizing a generalization into a principle or algebraic rule. This was apparent when Larissa wrote the rule  $s \cdot (\frac{2}{3}) = b$  after having searched for a relation between the rotations of a small and a large gear. Finally, there are times when a reflection generalization either expands beyond the situation that framed the generalizing action—as in the case of Juanita's global rule—or is not clearly tied to a discernible generalization action.

## Part 2: Using the Taxonomy to Account for the Evolution of Generalizing

The generalization taxonomy provides a way to consider which types of generalizations could represent more sophisticated knowledge than others. Analysis of the teaching experiment data revealed major trends in the growth of students' generalizing actions. Early in the sessions students focused on generalizing from immediate relationships between quantities, whereas in the later sessions they generalized across different quantitative situations to establish more global rules about linearity. In attempting to capture the nature of increased sophistication, students' later generalizations were contrasted to earlier generalizations. The more sophisticated generalizations were those that reflected a more complete, broad, and nuanced understanding of mathematical properties and situations.

Increased sophistication implies the development of generalizations that are more complex, inclusive, and refined than prior generalizations. Three criteria for

increased sophisticated proved useful to describe students' growth. The first criterion is movement from Type 1 (Relating) to Type 2 (Searching) to Type 3 (Extending) actions. As students moved from Relating to Searching to Extending, their actions grew more goal-oriented and creative. Students who engaged in relating actions made spontaneous connections; their associations were not necessarily deliberate or intentional. When searching, students anticipated the existence of a relationship of similarity and coordinated their efforts on locating it. Searching actions therefore represented deliberate problem-solving behavior, whereas relating actions did not. Finally, extending involved reasoning about objects that were not present. Generation of new knowledge occurred within extending in a way that did not occur when students engaged in relating or searching actions.

The second criterion is the generation of new inferences, which developed because repeated cycles of generalizing produced new understanding as well as the creation of new ideas. Through engaging in generalizing actions and producing reflection generalizations, students may develop fresh insight into a problem situation. Thus, if the evolution of a student's generalizations resulted in attention to a different aspect of a problem, the creation of previously undeveloped ideas or more inclusive connections, or the promotion of a more complex and nuanced understanding, then his or her generalizing activity was considered to have grown in sophistication.

The final criterion is the ability to support correct and powerful justifications. This emerged from the belief that generalizations for which students can provide appropriate and correct justifications will represent more sophisticated knowledge than those for which students cannot provide such arguments. Study participants who created generalizations that they could appropriately justify also demonstrated evidence that these generalizations were the ones that were well connected to other knowledge.

### *Iterative Action-Reflection Cycles of Generalization*

Identifying relations between generalizing actions and reflection generalizations revealed that study participants did not engage in isolated generalizing actions, produce associated reflection generalizations, and then move on. Instead, students moved between these behaviors, and each influenced the other in a way that allowed the participants to develop new knowledge. Thus, students engaged in iterative "action-reflection cycles." Although their initial generalizing actions and associated reflection generalizations may have been limited or incorrect, subsequent cycles built on previous attempts to develop more sophisticated generalizations.

The following episode is drawn from the teaching experiment, in which students made connections between situations, extended their reasoning, described patterns, and ultimately made inferences about quantitative relationships. Unlike the brief data examples seen thus far, the following episode is a longer, descriptive

excerpt presented to illustrate both the nature of students' action-reflection cycles of generalization and the ways in which students' generalizing can increase in sophistication.

The session began with the following problem: "The table (see Figure 5) shows some of the distances and times that Clown traveled. Is he going the same speed the whole time, or is he speeding up and slowing down? How can you tell?"

Julie: Hey, we had this before!

Larissa: Really? Really?

Julie: Wasn't this like the gears?

Larissa: Yes, it was!

The girls performed the generalizing action of connecting back to a prior situation, noting that this table was similar to a table with which they had worked in the gears situation. (In fact, it was the same numerical data set in a different context). Julie produced the reflection generalization of the identification of two situations as the same by asking, "wasn't this like the gears?" Then Larissa said to Julie, "this is the two thirds pair", thus identifying a common property across both problems. The first gear problem the students had encountered involved a gear ratio of two thirds. Larissa referred to this as the "two thirds pair" because she and the other students typically identified the smallest whole-number rotation pair for each gear set.

Larissa then made use of an idea she had developed when working with gears. She had previously created a gear-pair table in which the rotations of a small gear increased by one half, and the rotations of a large gear increased by one third. Now Larissa decided to do the same thing. She created a new table of values, in which the centimeters increased by one half and the seconds increased by one third. Thus, Larissa's reflection generalization is the influence of a prior idea in creating the table shown in Figure 14.

In creating the new table, Larissa extended her reasoning by continuing the "1/2:1/3" pattern in the original table. She explained that "I made the table to start at 1, 2:1/3"

cm	SEC
1	$\frac{2}{3}$
$1\frac{1}{2}$	1
2	$1\frac{1}{3}$
$2\frac{1}{2}$	$1\frac{2}{3}$
3	2
$3\frac{1}{2}$	$2\frac{1}{3}$
4	$2\frac{2}{3}$
$4\frac{1}{2}$	3

FIGURE 14 Larissa's newly created uniform table of distance and times.



and then I used the gear pair that we had, with the two thirds, um relationship, so I went up by one half on one side and two thirds on the other side.” Although Larissa said two thirds, she actually used one third in her table, so she could have misspoken. The fact that Larissa actually said “gear pair” is further evidence that she was thinking about the gear situation and implementing what she had learned there.

Timothy noticed that Larissa’s table represented a general pattern, which he stated explicitly: “They’re going up by halves and one thirds.” So his reflection generalization was the identification of the general pattern in Larissa’s table. The teacher/researcher asked Larissa what she noticed by making this table, and she said “That, the, since the pattern’s staying the same throughout the whole thing, he has to be going the same pace.” Larissa therefore used the pattern to make a deduction about the clown’s speed. To clarify, she also made a statement of a general pattern: “Since it’s always going up by one half on the centimeters side and two, and one third on the seconds side, he can’t be speeding up if, if he’s going the same, but they’re the same.” The teacher/researcher pushed Larissa to explain further:

TR: How come, how come the pattern staying the same means that he’s walking the same speed?

Julie: Oh! Oh! Oh!

Larissa: Because if he sped up, then the, the distance that he went in the amount of seconds that he went would decrease. And if he slowed down then it would increase.

Other students chimed in with their possible reasons why, and eventually the teacher/researcher asked for a summary:

TR: I think I know what you’re all saying. But let me just ask one final time to restate why increasing by  $1/2$  cm every time the seconds increases by one third, why that pattern holding means that the clown is walking the same speed?

Larissa: Because throughout the table it’s going, he’s going  $1/2$  cm in one third of a second. So every time he went another half a centimeter, he’d have to go another third of a second.

Larissa’s justification involved mentally holding one quantity constant and changing the other to deduce how that would affect the clown’s speed. Her statement also contained an identification of a continuing phenomenon (every time...) and, through the process of explaining, her focus shifted away from the naked numbers toward the idea of the clown walking with the associated quantities, centimeters and seconds.

Once Larissa and Julie had identified the speed and gear situations as similar, Larissa was able to identify a property for both cases, the “two thirds pair.” So Larissa had moved from a generalizing action, connecting back, to a reflection

generalization, the identification of a common property. After she identified this property, the influence of a prior idea is evident as Larissa created the new table. Thus, movement from one type of reflection generalization to another type occurred. This allowed Larissa to create something new, a table of speed values with previously unproduced same-speed pairs. As she did this, Larissa engaged in the generalizing action of extending by continuing the “1/2:2/3” pattern. By extending, Larissa created new mathematical objects in the form of new same-speed pairs.

Timothy and Larissa both identified the 1/2:1/3 general pattern, which spurred Larissa to conclude that the clown must be going the same pace. The idea that Larissa learned something new is indicated by her remarks when she was asked to explain why the pattern staying the same meant that the clown was walking the same speed. Specifically, Larissa’s explanation connected the number patterns in the table to the quantities and the quantitative relationships in the situation, a connection she previously lacked. Larissa made inferences about those quantitative relationships thanks to her generalizing acts with the numbers and number patterns.

The episode demonstrates that students can generalize in one way and then, through their actions and statements, be spurred to generalize in a new way. The chain of students’ generalizing did not occur haphazardly, but instead developed in ways that allowed the students to build more sophisticated connections. In each case, Larissa’s generalizing action sparked an idea that she was able to describe as a reflection generalization. By making her reflection generalizations explicit, Larissa was able to progress to a higher level of generalizing action, fulfilling the first criterion of moving from Type 1 (Relating) to Type 2 (Searching) to Type 3 (Extending). Each level of generalizing in turn encouraged new ideas and connections, which is described by the second criterion, the generation of new inferences. The hypothesis that Larissa developed new knowledge is additionally supported by her justification connecting number patterns to quantities, which fulfills the third criterion requiring correct and powerful justifications.

## DISCUSSION

Prior research on both transfer and generalization has distinguished different types of generalization (e.g., Davydov, 1990; Detterman, 1993), described levels of generalization and categories of transfer (e.g., Barnett & Ceci, 2002; Garcia-Cruz & Martinon, 1998), and illustrated strategies for developing generalizations (Lannin, 2003). However, none of the existing constructs presents a comprehensive yet empirically grounded categorization system. The Generalization Taxonomy fills this gap by combining the dimensions raised by other constructs into one comprehensive system. Moreover, the Generalization Taxonomy provides a number of con-

nections between mathematics-education generalization research and transfer research. This section (a) discusses the connections between these two traditions of generalization research, (b) addresses the implications of having adopted an actor-oriented perspective toward mathematical generalization, and (c) considers some methodological contributions gained by reconceptualizing what it means to generalize.

### Connecting Two Traditions of Generalization Research

The Generalization Taxonomy not only builds on constructs identified in the transfer literature and the mathematics-education generalization literature, but also extends both in a number of ways. Specifically, several categories within the taxonomy are related to constructs identified in both the traditional and reconceived transfer literature. For instance, the generalizing action of relating includes the action of forming an association of similarity between two or more situations, which shares some elements with Carraher and Schliemann's (2002) discussion of the ways in which learners can adjust their understanding of initial learning situations to create relations of similarity with transfer situations. Some extending actions could constitute cases of vertical transfer (Gagné, 1977), particularly during times when students may remove particulars or expand the range of applicability to develop higher level understanding. The reflection generalization category of influence mirrors several reconceived views of transfer as the influence of prior learning on new situations (Lobato, 2006; Marton, 2006).

Several categories within the taxonomy are also related to constructs identified in the mathematics education generalization literature. For instance, when engaged in the generalizing action of relating, students sometimes focus on properties that they see as similar across two situations, which coincides with Dreyfus's (1991) notion of generalizing as identifying commonalities. Extending actions intersect with Kaput's (1999) idea of extending the range of reasoning beyond the case or cases considered, and Harel's (2001) notions of process-pattern generalization and result-pattern generalization occur when students extend by operating or by continuing a pattern. The reflection generalization category of the identification of general rules is similar to Lannin's (2003) contextual strategy for developing a generalization, in which students construct a rule on the basis of a relation determined from the problem. These connections between students' generalizing actions, reflection generalizations, and views of transfer support the notion that certain types of mathematical generalization can be conceptualized as a form of transfer.

However, the Generalization Taxonomy also extends beyond existing types of generalization identified in either the transfer literature or the mathematics education literature. For instance, the taxonomy offers a description of both the types of mental processes students engage in as they generalize and the statements of gen-

eralization students produce. Although others have described either student actions while generalizing (Dubinsky, 1991) or final statements of generalization (Stacey & MacGregor, 1997), only one other construct attends to the difference between generalizing acts and products of generalization. Namely, Davydov's (1990) work discusses the difference between generalizing processes and products, but this distinction depends on a more specific definition of generalization as the identification of a class of things, all sharing a particular property. Although Davydov's notion of process is similar to the generalizing action of extending, his notion of product refers to a student's ability to abstract himself or herself from certain particular and varying attributes of an object. This could include many reflection generalizations, such as the statement of a general pattern or a global rule.

By elaborating the previously ignored action–reflection distinction, the Generalization Taxonomy can help researchers discern how these types are related to one another. The data presented in this study suggest a connection between the ways in which students engage with a problem, and the resulting generalization statements they produce. Through an examination of which generalizing actions promote particular reflection generalizations, the taxonomy allows one to study the processes students go through as they generalize, beginning with initial mental acts and culminating in statements of generalization.

Finally, although existing constructs describing “the explicit identification and exposition of commonality across cases as a generalizing activity” (Kaput, 1999) fall under the umbrella of the generalizing action of searching, no existing research distinguishes the different ways in which one can identify and expose those commonalities. The searching category in the Generalization Taxonomy, which includes acts of students attending to what was the same across different situations, cases, numbers, or problems, accounts for these differences by describing four distinct searching actions.

### Benefits of an Actor-Oriented Generalization Taxonomy

As discussed in the introduction, one common limitation of the traditional transfer approach and much of the literature on generalization in mathematics education has been a reliance on the observer's perspective. This has led to a smaller range of student actions that are capable of being measured as generalizations, which has resulted in an underestimation of the amount of generalizing that occurs. Furthermore, the categorization systems developed within both traditions reflect this perspective. For instance, Barnett and Ceci (2002) created a transfer taxonomy with nine dimensions that contributes to the literature by clarifying distinctions between near and far transfer; however, its observer's orientation results in the taxonomy describing task dimensions rather than students' generalizing processes. Although Lobato's (2003) actor-oriented transfer perspective addresses these limitations, it does not offer any type of system for categorizing generalizations. Thus, one con-

tribution of this study to the on going transfer dialog is a taxonomy that allows researchers to identify types of generalizing actions (and the associated reflection generalizations) by attending to students' perceptions of similarity.

### *Learners' Conceptions*

By adopting the actor-oriented transfer perspective to incorporate multiple dimensions into a single categorization scheme, the Generalization Taxonomy also provides a way to identify novice students' actions throughout the evolution of their abilities to generalize. Bransford and Schwartz (1999) warned that

Prevailing theories and methods of measuring transfer work well for studying full-blown expertise, but they represent too blunt an instrument for studying the smaller changes in learning that lead to the development of expertise. New theories and measurements of transfer are required. (p. 24)

By identifying actions and reflections that represent a range of sophistication, the taxonomy allows researchers to study novices' conceptions as they generalize.

Moreover, an adoption of the actor-oriented perspective allows for an expansion of the range of student actions that "count" as generalization. For instance, although the generalizing actions that mirror work on transfer fall within the relating or extending categories, searching actions are not those which researchers have typically considered to constitute generalization. Data from the study revealed that students frequently engaged with problems in ways that originally appeared to be unproductive, but later were able to develop powerful results. For instance, a student may initially approach a problem by engaging in the generalizing action of searching for the same pattern or relationship in a table. This search may at times constitute an extended, inefficient process until an appropriate pattern is identified. However, if allowed to continue this action, searching can help a student ultimately formulate and extend algebraically useful relations. The Generalization Taxonomy validates and elaborates searching actions by demonstrating how these actions can lead to the production of the types of generalizations valued by the teaching and research communities.

### *The Evolution of Generalizing*

The Generalization Taxonomy's actor-oriented perspective also allows for the study of the evolution of students' reasoning. It accounts for how students go through cycles of generalizing, and how these cycles can lead to more sophisticated and more powerful general statements over time. Most important, the taxonomy moves beyond casting generalization as an activity that students either do or do not engage in successfully to allow researchers to identify what students see as general. This move away from a success-failure model toward a more nuanced

view of generalizing could help researchers and teachers better understand the processes students engage in as they generalize, either correctly or incorrectly. Furthermore, teachers who can view incomplete or incorrect generalizations as necessary steps in the larger process of developing a habit of generalizing will be more likely to support those steps, rather than curtail such activity. By adopting the actor-oriented transfer lens, the Generalization Taxonomy allows the researcher to move beyond the standard model of correct pattern identification, formalization, and rule development as the sole route toward mathematical generalization.

### Methodological Contributions

A final benefit of borrowing from the insights gained in the reconceived transfer literature rests with the study's shift beyond the traditional methodological orientation. Traditional studies examining both transfer and generalization rely on discrete tasks, either the learning task–transfer task model in transfer studies, or the pattern generalization task model in generalization studies. The Generalization Taxonomy introduces a way for researchers to study processes of generalization and transfer in a different way; it provides a structure for identifying processes of generalization within one classroom setting and even within one problem context. Moreover, the Generalization Taxonomy addresses students' actions as they occur in instructional settings over longer periods of time. By studying students' emerging abilities to seek similarity, extend their reasoning, make connections between ideas and situations, and ultimately develop more general structures, the study reported in this article was able to capture students engaging in generalizing activities that would have been otherwise missed.

### Further Questions

The Generalization Taxonomy represents only what occurred in two settings with seventh- and eighth-graders studying linear functions. The small-scale nature of the project cannot give rise to a definitive scheme, but it can offer an initial system with which to interpret students' reasoning in other settings. Sloane and Gorard (2003) describe three main stages of model building: model formulation, estimation or fit, and model validation. This study supported the first stage of model building, the formulation of the taxonomy as a model for students' generalizations. Although Sloane and Gorard emphasize that "model formulation is often the most important and the most difficult stage of the research process" (p. 29), it is still necessary to further test the taxonomy across different populations and content domains. Thus, future studies should focus on the testing and validation of the taxonomy with new data sets.

Furthermore, the results presented here represent the influence of a small number of factors on students' generalizing, such as the types of problems students en-

countered, the types of artifacts used, the nature of the questions the students responded to, and how the students focused their attention as they worked with problems. Other factors surely influenced students' generalizing activity, such as the classroom culture in which they worked, the discourse patterns in which the students engaged, the students' background knowledge and attitudes toward the material they encountered, and the ways in which the students interacted with their peers and their teacher. Future work geared toward better understanding how instructional environments support generalizing will contribute to a richer framework that takes into account these larger social influences on students' generalizing.

### ACKNOWLEDGMENTS

This research was supported by the National Science Foundation (NSF) under Grant REC-9733924. The views expressed do not necessarily reflect official positions of the NSF.

I thank Eric Knuth, Mitchell Nathan, and Patrick Callahan for their thoughtful comments on a draft of this article. After submitting this article, I received constructive suggestions from four anonymous reviewers and from the editors: Janet Kolodner, Yasmin Kafai, and Joanne Lobato.

### REFERENCES

- Barnett, S., & Ceci, S. J. (2002). When and where do we apply what we learn? A taxonomy for far transfer. *Psychological Bulletin*, 128(4), 612-637.
- Beach, K. (1999). Consequential transitions: A sociocultural expedition beyond transfer in education. *Review of Research in Education*, 28, 46-69.
- Bernard, H. E. (1988). *Research methods in cultural anthropology*. Beverly Hills, CA: Sage.
- Bransford, J. D., & Schwartz, D. L. (1999). Rethinking transfer: A simple proposal with multiple implications. In A. Iran-Nejad & P. D. Pearson (Eds.), *Review of Research in Education* (Vol. 24, pp. 61-100). Washington, DC: American Educational Research Association.
- Brown, J. S., Collins, A., & Duguid, P. (1989). Situated cognition and the culture of learning. *Educational Researcher*, 18, 32-41.
- Campione, J. C., Shapiro, A. M., & Brown, A. L. (1995). Forms of transfer in a community of learners: Flexible learning and understanding. In A. McKeough, J. Lupart, & A. Marini (Eds.), *Teaching for transfer: Fostering generalization in learning* (pp. 35-68). Mahwah, NJ: Lawrence Erlbaum Associates, Inc.
- Carpenter, T., & Franke, M. (2001). Developing algebraic reasoning in the elementary school: Generalization and proof. In H. Chick, K. Stacey, J. Vincent, & J. Vincent (Eds.), *Proceedings of the 12th ICMI study conference: The Future of the teaching and learning of algebra* (pp. 155-162). Melbourne, Australia: The University of Melbourne.
- Carraher, D., & Schliemann, A. D. (2002). The transfer dilemma. *The Journal of the Learning Sciences*, 11, 1-24.



- Cobb, P., & Steffe, L. P. (1983). The constructivist researcher as teacher and model builder. *Journal for Research in Mathematics Education*, 28, 258–277.
- Davydov, D. D. (1990). *Soviet studies in mathematics education: Vol. 2. Types of generalization in instruction: Logical and psychological problems in the structuring of school curricula*. Reston, VA: National Council of Teachers of Mathematics.
- Detterman, D. (1993). The Case for the Prosecution: Transfer as an epiphenomenon. In D. Detterman & R. Sternberg (Eds.), *Transfer on Trial: Intelligence, cognition, and instruction*. Norwood, NJ: Ablex.
- diSessa, A. A., & Wagner, J. F. (2005). What coordination has to say about transfer. In J. P. Mestre (Ed.), *Transfer of learning from a modern multidisciplinary perspective* (pp. 121–154). Greenwich, CT: Information Age Publishing.
- Dreyfus, T. (1991). Advanced mathematical thinking processes. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 25–41). Dordrecht, the Netherlands: Kluwer.
- Dubinsky, E. (1991). Reflective abstraction in advanced mathematical thinking. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 95–123). Dordrecht, the Netherlands: Kluwer.
- English, L., & Warren, E. (1995). General reasoning processes and elementary algebraic understanding: Implications for instruction. *Focus on Learning Problems in Mathematics*, 17(4), 1–19.
- Gagné, Robert M. (1977). *The conditions of learning and theory of instruction*. New York: Holt, Rinehart & Winston.
- Garcia-Cruz, J., & Martinon, A. (1997). Actions and invariant in schemata in linear generalizing problems. In E. Pehkonen (Ed.), *Proceedings of the 21st Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 289–296). Lahti, Finland: PME.
- Glaser, B. G., & Strauss, A. L. (1967). *The discovery of grounded theory: Strategies for qualitative research*. Hawthorne, NY: Aldine.
- Greeno, J. G. (1997). Response: On claims that answer the wrong questions. *Educational Researcher*, 26(1), 5–17.
- Gruber, H., Law, L., Mandl, H., & Renkl, A., (1996). Situated learning and transfer. In P. Reimann & H. Spada (Eds.), *Learning in humans and machines: Towards an interdisciplinary learning science* (pp. 168–188). Oxford, England: Pergamon.
- Harel, G. (2001). The development of mathematical induction as a proof scheme: A Model for DNR-based instruction. In S. Campbell & R. Zaskis (Eds.), *Learning and teaching number theory* (pp. 185–212). Norwood, NJ: Ablex.
- Harel, G., & Tall, D. (1991). The general, the abstract, and the generic. *For the Learning of Mathematics*, 11, 38–42.
- Kaput, J. (1999). Teaching and learning a new algebra with understanding. In E. Fennema, & T. Romberg (Eds.), *Mathematics classrooms that promote understanding* (pp. 133–155). Mahwah, NJ: Lawrence Erlbaum Associates, Inc.
- Kieran, C. (1992). The learning and teaching of school algebra. In D. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning* (pp. 390–419). New York: Macmillan.
- Krutetskii, V. A. (1976). *The psychology of mathematical abilities in schoolchildren, Volume 1*. Chicago: University of Chicago Press.
- Lannin, J. K. (2003). Developing algebraic reasoning through generalization. *Mathematics Teaching in the Middle School*, 8(7), 342.
- Lappan, G., Fey, J., Fitzgerald, W., Friel, S., & Phillips, E. (1998). *Connected mathematics project*. Ann Arbor, MI: Dale Seymour Publications.
- Larson, R., Boswell, L., Kanold, T., & Stiff, L. (2001). *Algebra I*. Evanston, IL: McDougal Little.
- Lave, J. (1988). *Cognition in practice: Mind, mathematics, and culture in everyday life*. Cambridge, UK: Cambridge University Press.
- Lee, L. (1996). An initiation into algebraic culture through generalization activities. In N. Bednarz, C. Kieran, & L. Lee (Eds.), *Approaches to algebra* (pp. 87–106). Dordrecht, the Netherlands: Kluwer.



- Lobato, J. (2003). How design experiments can inform a rethinking of transfer and vice versa. *Educational Researcher*, 32(1), 17–20.
- Lobato, J. (2006). Alternative perspectives on the transfer of learning: History, issues, and challenges for future research. *Journal of the Learning Sciences*, 15(4), ???.
- Lobato, J. (in press-a). How rethinking assumptions about the “transfer” of learning can inform research, instructional practices, and assessment. In C. Rasmussen & M. Carlson (Eds.), *Making the Connection: Research and Teaching in Undergraduate Mathematics*. Washington, DC: Mathematical Association of America.
- Lobato, J. (in press-b). Research methods for alternative approaches to transfer: Implications for design experiments In A. Kelly & R. Lesh (Eds.), *Design Research in Education*. Mahwah, NJ: Lawrence Erlbaum Associates, Inc.
- Lobato, J., Clarke, D., & Ellis, A. B. (2005). Initiating and eliciting in teaching: A reformulation of telling. *Journal for Research in Mathematics Education*, 36(2), 101–136.
- Lobato, J., & Siebert, D. (2002). Quantitative reasoning in a reconceived view of transfer. *The Journal of Mathematical Behavior*, 21(1), 87–116.
- Lobato, J., & Thanheiser, E. (2000). Using technology to promote and examine students’ construction of ratio-as-measure. In M. L. Fernández (Ed.), *Proceedings of the 22nd annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 371–377). Columbus, OH: Educational Resources Information Center.
- Lobato, J., & Thanheiser, E. (2002). Developing understanding of ratio and measure as a foundation for slope. In B. Litwiller & G. Bright (Eds.), *Making sense of fractions, ratios, and proportions: 2002 yearbook* (pp. 162–175). Reston, VA: National Council of Teachers of Mathematics.
- Marton, F. (2006). Sameness and difference in transfer. *Journal of the Learning Sciences*, 15(4), ???.
- Orton, A., & Orton, J. (1994). Students’ perception and use of pattern and generalization. In J. P. da Ponto & J. F. Matos (Eds.), *Proceedings of the 18th international conference for the Psychology of Mathematics Education* (Vol. 3, pp. 407–414). Lisbon, Portugal: PME Program Committee.
- Pea, R. (1989). *Socializing the knowledge transfer problems* (Report No. IRL89-0009). Palo Alto, CA: Institute for Research on Learning.
- Perkins, D. N., & Salomon, G. (1988). Teaching for transfer. *Educational Leadership*, 46(1), 22–32.
- Perkins, D. N., & Salomon, G. (1989). Are Cognitive Skills Context Bound? *Educational Researcher*, 18(1), 16–25.
- Piaget, J., & Henriques, G. (1978). *Recherches sur la généralisation*. Paris: Presses Universitaires de France.
- Reed, S. K., Ernst, G. W., & Banerji, R. (1974). The role of analogy in transfer between problem states. *Cognitive Psychology* 6, 436–450.
- Roschelle, J., & Kaput, J. J. (1996). SimCalc Mathworlds for the Mathematics of Change. *Communications of the ACM*, 39(8), 97–99.
- Singley, M. K., & Anderson, J. R. (1989). *The transfer of cognitive skill*. MA: Cambridge University Press.
- Sloane, F., & Gorard, S. (2003). Exploring modeling aspects of Design Experiments. *Educational Researcher*, 32(1), 29–31.
- Stacey, K. (1989). Finding and using patterns in linear generalising problems. *Educational Studies in Mathematics*, 20, 147–164.
- Stacey, K., & MacGregor, M. (1997). Building foundations for algebra. *Mathematics Teaching in the Middle School*, 2(4), 253–260.
- Strauss, A., & Corbin, C. (1990). *Basics of qualitative research: Grounded theory procedures and techniques*. Newbury Park, CA: Sage.
- Thompson, P. W. (1994). The development of the concept of speed and its relationship to concepts of rate. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 181–234). Albany, NY: SUNY Press.

Tuomi-Gröhn, T., & Engeström, Y. (2003). *Between school and work: New perspectives on transfer and boundary crossing*. Amsterdam: Pergamon.

Von Glasersfeld, E. (1995). *Radical constructivism: A way of knowing and learning*. Bristol, PA: Falmer.

Copyright of *Journal of the Learning Sciences* is the property of Lawrence Erlbaum Associates and its content may not be copied or emailed to multiple sites or posted to a listserv without the copyright holder's express written permission. However, users may print, download, or email articles for individual use.