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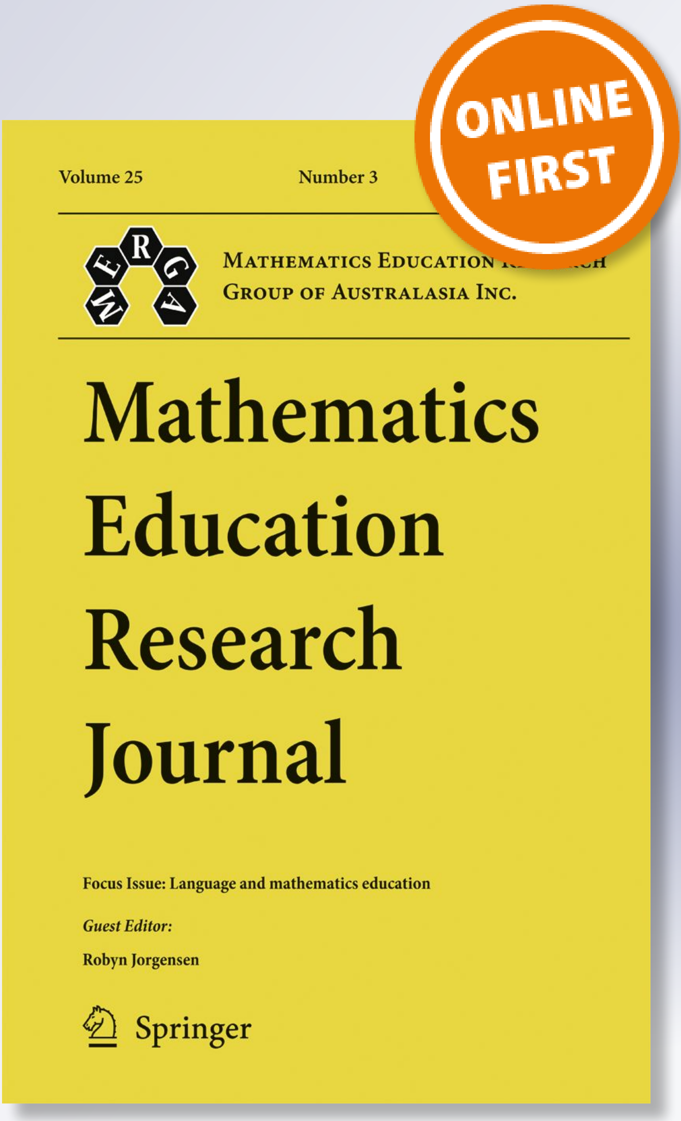

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ORIGINAL ARTICLE

Teacher moves for supporting student reasoning

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Abstract Teachers play a critical role in supporting students' mathematical engagement. There is evidence that meaningful student engagement occurs more often in student-centered classrooms, in which the teacher and the students mutually share mathematical authority. However, teacher-centered instruction continues to dominate classroom discourse, and teachers struggle to effectively support student inquiry. This paper presents a framework of teacher moves specific to inquiry-oriented instruction, the *Teacher Moves for Supporting Student Reasoning* (TMSSR) framework. Based on the analysis of four instructors' implementations of a middle grades (ages 12–14) research-based unit on ratio and linear functions, the TMSSR framework organizes pedagogical moves into four categories, *eliciting*, *responding*, *facilitating*, and *extending*, and then places individual moves within each category on a continuum according to their potential for supporting student reasoning. In this manner, the TMSSR framework characterizes how multiple teacher moves can work together to foster an inquiry-oriented environment. We detail the framework with data examples and then present a classroom episode exemplifying the framework's operation.

Keywords Mathematics teacher education · Classroom discussion · Teacher moves · Algebra

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Introduction

To effectively support meaningful student learning, classroom discussions must focus both on important mathematical ideas and on students' development of mathematical meaning through communicative processes (Leikin and Dinur 2007). The teacher's role in these discussions is central in promoting students' understanding; s/he must choose appropriate tasks, determine when and how to foster students' thinking, and decide when to allow students to engage in productive struggle (Rittenhouse 1998). Meaningful student understanding is more likely to occur in classrooms in which the students and the teacher mutually share mathematical authority (e.g., Knuth and Peressini 2001; Wertsch and Toma 1995; Wood 1998). Within this model of instruction, students take on intellectual responsibility for constructing and defending their own mathematical ideas, which has been shown to be positively related to student achievement outcomes, even when controlling for prior achievement (Webb and Palincsar 1996). This instruction model requires the teacher to make many on-the-spot strategic decisions about how to probe and support students' thinking. Merely increasing the quantity of student talk is not sufficient; students require spaces in which they can explore ideas, develop conjectures, make connections, and justify their thinking (Hunter 2012; Nathan and Knuth 2003). How teachers promote productive student discourse has been an important research focus for the past 10 years (Hunter et al. 2016). Therefore, there is an increased focus on inquiry-oriented classrooms in which students are positioned as problem solvers (Leach et al. 2014) and student reasoning is central to classroom activity (Hunter 2008).

Despite the affordances of shifting to student-centered discussions, teacher-centered instruction continues to dominate mathematics classrooms (Cuban 1993; Herbel-Eisenmann et al. 2013). Students experience few opportunities to share their ideas or ask questions (Cazden 2001; Graesser and Person 1994), and student participation is often relegated to responses to teachers' queries about facts and procedures (Franke et al. 2009). Although teachers may begin by asking open-ended questions, classroom discourse then often shifts to a more traditional initiate-response-evaluate pattern (IRE), revealing the challenges teachers experience in moving from a transmission style of instruction to one that fosters open-ended participation (Cady et al. 2006; Larsson and Ryve 2012; Leikin and Rosa 2006; Truxaw and DeFranco 2008). This body of work emphasizes the need to provide teachers with explicit and ongoing support in reflecting on and changing their instructional practices.

Despite these challenges, there are cases depicted in the literature in which teachers do effectively support student inquiry (e.g., Frailvillig et al. 1999; Hufferd-Ackles et al. 2004; Leatham et al. 2015; Smith and Stein 2011; Staples 2007). These studies show that with proper support, teachers can establish classroom norms that foster inquiry-based exploration of rich tasks (Sullivan et al. 2013) and shift classroom discourse to a model emphasizing student contributions and open-ended discussions. The goal of this study was to develop a framework of teacher moves specific to inquiry-oriented instruction, in which teachers enacted research-based units in middle-grades (ages 12–14) algebra classrooms that were developed to support student *reasoning*. We present results from the implementation of a research-based unit on ratio and linear function across four instructors, which yielded the TMSSR framework. We then highlight an episode that demonstrates how the framework operates in a classroom discussion.

Reasoning has been addressed in a number of different ways in the research literature. The Australian Curriculum (Australian Curriculum, Assessment and Reporting Authority [ACARA], 2015) identifies reasoning as one of the four key proficiencies, involving "logical thought and actions, such as analyzing, proving, evaluating, explaining, inferring, justifying, and generalising" (p. 2). Researchers have described mathematical reasoning in terms of conjecturing, generalizing, and justifying (Lannin et al. 2011; Russell 1999), discovering new ideas or concepts (Ball and Bass 2003; Mata-Pereira and da Ponte 2017), and engaging in the processes of induction, deduction, and abduction (Herbert 2014; Steen 1999; Yackel and Hanna 2003).

We draw on the recent work of Jeannotte and Kieran (2017), who conducted a meta-analysis elaborating a conceptual model of mathematical reasoning for the teaching and learning of school mathematics. Jeannotte and Kieran's (2017) model articulates a structural aspect and a process aspect of reasoning. The structural aspect addresses induction, abduction, and deduction, while the process aspect characterizes the various processes associated with structure of mathematical reasoning. These processes include (a) searching for similarities or differences (i.e., generalizing, conjecturing, pattern identification, comparing, and classifying), (b) validating (i.e., justifying and proving), and (c) exemplifying as a way of supporting the first two processes. Thus, following Jeannote and Kieran's model, we define mathematical reasoning as a process of inference that includes searching for similarities or differences, validating, and exemplifying.

Review of the literature: teacher moves and frameworks for pedagogical moves

There is a rich and varied literature on teachers' questioning and classroom discourse patterns. Studies addressing teacher moves identify individual verbal acts that fall along a range of effectiveness in supporting student reasoning and engagement and discuss the differences between less-productive and more-productive questioning and discourse patterns. There are also studies specifically addressing the types of teacher questioning that support student thinking, justification, or classroom participation. Further, a sub-set of the teacher moves literature organizes patterns of interaction or teacher acts into *frameworks*; these frameworks address teacher moves from a variety of perspectives, such as supporting collaborative inquiry, fostering conceptual understanding, or building on student thinking. In the following section, we distinguish between studies that articulate particular teacher moves, and those that offer frameworks of moves. We then address how the TMSSR framework offers a new perspective by broadening the teacher moves under consideration beyond types of questioning, and by placing a constellation of moves on a continuum in relation to one another. In this manner, the TMSSR framework addresses both the relationships between particular moves and the ways in which moves can operate together in order to support student reasoning.

Teacher moves

Researchers have identified a number of teacher moves that characterize instructional activity in mathematics classrooms. The majority of these studies can be characterized as identifying either (a) individual verbal acts, (b) dichotomies of moves, or (c) types of moves that support reasoning and justification. Individual verbal acts are those taken by teachers, such as re-voicing (the re-utterance of another's speech through repetition, expansion, or rephrasing, Forman et al. 1998), eliciting (determining what students know and how they think, Frailvillig et al. 1999; Lampert et al. 2013; Staples 2007), and extending (allowing students to further develop connections among ideas, Cengiz et al. 2011; Frailvillig et al. 1999). These studies characterize a variety of teachers' pedagogical actions, including moves that are more and less effective for promoting student reasoning.

Dichotomies of moves typically address distinctions between more and less productive patterns of interactional activity. For instance, Wood (1998) distinguished between the funneling pattern (Bauersfeld 1980), in which an incorrect answer is a starting point from which a teacher leads a student through a series of increasingly explicit questions to pull out a correct answer, and the focusing pattern, in which a teacher's questions serve as an attempt to orient conversation to one aspect of a student's thinking. In another example, Voigt (1995) distinguished elicitation patterns from discussion patterns. In the former, the teacher proposes a task, students offer their solutions and strategies, and if the students' contributions are too divergent, the teacher poses leading questions to funnel the discussion towards the desired solution path. In the latter, students share a variety of solution strategies, with the teacher contributing with additional suggestions, reformulations, and questions in order to facilitate the evolution of a joint solution. Knuth and Peressini (2001) drew a similar distinction between univocal and dialogic discourse. Univocal discourse is characterized by the authority given to the teacher to evaluate all contributions, whereas dialogic discourse is characterized by sharing authority across all of the interlocutors. Truxaw and colleagues (2008) then built on this work to develop models of teaching that promote discourse on a continuum from univocal to dialogic. This body of work depicts a variety of dichotomous moves that differ in their potential to foster student reasoning, with research emphasizing the utility of powerful moves such as revoicing, eliciting, responding, and extending as salient components of ambitious teaching (Cengiz et al. 2011; Herbel-Eisenmann et al. 2009; Lampert et al. 2013; Staples 2007).

Researchers have also sought to understand the type of pedagogical activity that can support students' mathematical reasoning and justification practices. For instance, Wood (1994, 1998) described patterns of interaction in which teachers emphasized and validated important ideas present in students' responses rather than pre-determined solution methods. In a similar line of work, Staples (2007) offered a model capturing the critical work teachers do in supporting students' participation in collaborative inquiry practices. Lampert et al. (2013) also studied how teachers can support students' reasoning, identifying critical components of ambitious teaching. These moves include eliciting, interpreting, responding to student work, and attending to the details of student thinking. Franke et al. (2009) similarly detailed teachers' discourse related to helping their students develop complete and correct explanations, finding that effective teachers consistently pushed students to explain and share their thinking. Taken as a whole, these studies identify positive moves teachers can make, and indicate that students learn mathematics with greater understanding when they are allowed to explore, reason, and communicate about their ideas (Wood 1998).

Frameworks for pedagogical moves

A number of researchers have organized related teacher moves into frameworks identifying aspects of inquiry-oriented instruction. This literature includes frameworks that address sets of moves analyzed from a particular lens, such as discourse, or towards a specific goal, such as supporting collaborative inquiry. For instance, Herbel-Eisenmann et al. (2013) introduced the *Talk Moves* framework, which includes discourse moves such as waiting, inviting student participation, revoicing, asking students to revoice, probing students' thinking, and creating opportunities for students to engage with one another's reasoning. Krussel et al. (2004) also described a framework of discourse moves, which includes the actions teachers can take to mediate, participate in, and influence the nature of mathematical talk in a classroom setting, and Staples (2007) offered a model capturing teachers' work in supporting students' participation in collaborative inquiry practices. To support the development of inquiry communities, Hunter (2008) developed a framework of communication and participation moves supporting progressively more proficient mathematical practices (e.g., making conceptual explanations, generalizations, justifications). Finally, Leikin and Dinur (2007) characterized situations in which teachers managed classroom discussions in order to analyze factors shaping their flexibility in instruction.

Other frameworks attend to mathematical meaning making and building on students' ideas. For instance, Smith and Stein's (2011) work on orchestrating classroom discussions emphasizes the importance of building on student thinking as the basis for discussions. Drawing on that work, Leatham et al. (2015) then developed a framework called Mathematically Significant Pedagogical Opportunities to Build on Student Thinking (MOSTs), which is a tool to analyze the mathematical and pedagogical potential of student thinking that emerges during instruction. Another example of a framework emphasizing mathematical meaning making is Frailvillig et al. (1999) Advancing Children's Thinking (ACT) framework aimed at supporting conceptual understanding, which identifies strategies according to their primary functions and demonstrates how teacher moves can serve multiple goals. Cengiz et al. (2011) then extended this work to develop the Extending Student Thinking (EST) framework, which addresses the instructional actions teachers implement during extending episodes, such as encouraging reflection, reasoning, and justification.

The majority of the above-described frameworks characterize forms of discourse, highlight interactional patterns, or classify pedagogical strategies instrumental in fostering student-centered inquiry, but do not address shifts in teacher moves. One framework by Hufferd-Ackles et al. (2004) does address the way a classroom can shift from being teacher-focused to student-centered. This framework elaborates the development of a math-talk learning community, and the authors describe four levels across four dimensions in which a classroom can make this shift: Questioning, explaining mathematical thinking, source of mathematical ideas, and responsibility for learning.

The Hufferd-Ackles framework addresses an important shift towards a particular goal, namely, increasing and promoting student engagement and participation, along with transitioning mathematical authority and responsibility from the teacher to the students.

The Teacher Moves for Supporting Student Reasoning (TMSSR) framework differs from existing frameworks in a number of ways. Firstly, it is specific to mathematics, and was developed in order to address how teachers can foster mathematical reasoning in particular. Although some of the above frameworks, such as MOSTs, are about mathematically significant teacher moves, others are more general. In addition, the TMSSR framework organizes pedagogical moves into categories according to their potential for supporting student reasoning. The TMSSR framework also places teacher moves on a continuum. The continuum is not strictly hierarchical; instead, it organizes a constellation of teacher moves in relation to one another to characterize the ways in which these moves can *work together* to foster an inquiry-oriented environment; a characteristic unique among teacher move frameworks. Finally, the TMSSR framework goes beyond teacher questioning and discourse to include other practices that can foster student reasoning, such as re-representing, figuring out student reasoning, or providing guidance.

By addressing both relationships between moves and the ways in which multiple moves can operate together, the TMSSR framework offers a more comprehensive model of the ways in which teachers can support students' engagement in conceptually rich mathematics. In particular, we address the following questions: (1) What types of teacher moves occur in an inquiry-oriented algebra setting? (2) How can we characterize the nature of teacher moves in classroom instruction, and what are the relationships between these moves? (3) What is the potential support of the different pedagogical moves for fostering student reasoning?

Theoretical framework

We consider teachers' actions to be mediated by their beliefs about their students as learners of mathematics, by the students they teach, by the instructional tasks and manipulatives in use, and by their own mathematical goals for the tasks and the unit. In order to account for these influences, we draw from the strand of activity theory developed by Engeström (1987, 1999). This strand takes a multi-faceted approach to investigate the activities in which people are engaged, and acknowledges that activities are mediated by people's experiences, how they use tools and transform tools through activity, and by the context of the activity.

An activity consists of a subject, object, and actions. The *subject* is the person engaged in an activity. The *object* motivates the activity and gives it a specific direction. A*ctions* are "goal-directed processes that must be undertaken to fulfill the object" (Nardi 1996, p. 37). Thus, for the purposes of our study, the object is the moves teachers make to support student reasoning, and their actions are the processes they undergo to implement instruction. Therefore, actions are the ways in which teachers engage with the tasks prior to implementation and their planned instructional supports, such as modifications they make to tasks, how they plan to present a task, and how they intend to make use of manipulatives. Figure 1 provides a model of an activity system in this study. Tools in this activity system include the original tasks, teacher-researcher reflections implementing the tasks, and the gear manipulatives (plastic gears containing 8, 12, and 16 teeth). The activity theory lens enables us to view teachers' moves for supporting student reasoning as mediated by several factors, including students' engagement with tasks, classroom and mathematical norms, and teachers' beliefs and goals. Although two teachers may implement the same move, the effect on student reasoning may differ due to other mediating factors in the system.

Methods

The development of the TMSSR framework occurred as part of a larger study investigating the influence of quantitative reasoning on students' understanding of function (Ellis et al. 2016; Ellis et al. 2015; Reiten et al. 2015). We collected data from four implementations of a research-based unit on ratio and linear functions, the first two in teaching-experiment settings taught by teacher-researchers, and the second two in whole-classroom implementations of the same unit, taught by practicing middle-grades teachers. Below we first describe the instructional unit itself, as well as the supports for enacting the unit in practicing teachers' classrooms. We then provide details about each of the four unit implementations. We conclude with a section on the rounds of data analysis that led to the final version of the TMSSR framework.

The instructional unit

Our research team initially developed and refined the linear functions unit across several teaching experiments. An original aim of the unit was to foster students' generalizing and justifying activities, grounded in an exploration of linear function

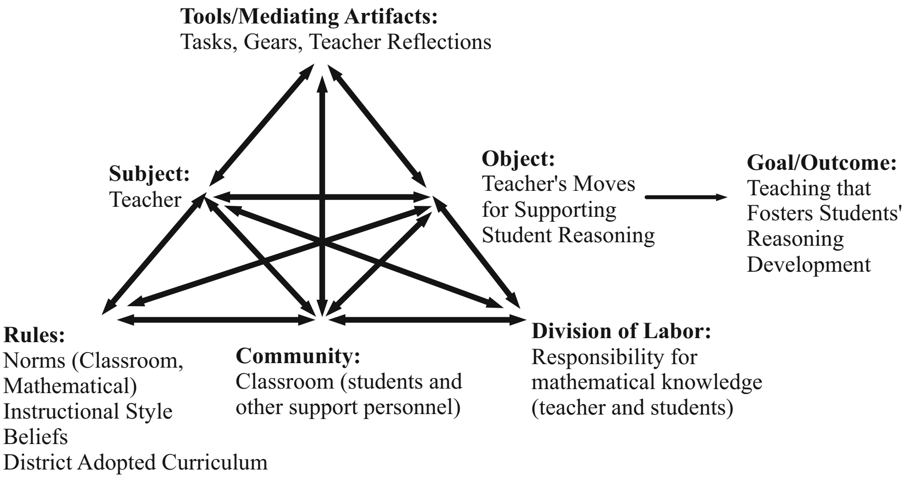

Fig. 1 Model of an activity system in this study (adapted from Engeström 1987, 1999)

within the context of gear ratios (see Ellis 2007). Once we had a stable set of activities, we conducted a series of professional-development sessions with practicing middle-school teachers who were interested in implementing a quantities-based function unit in their own classroom. We formalized the activities into a stand-alone unit consisting of a set of tasks, mathematical goals, and suggested modifications and extensions; these components of the unit constitute *tools* in the activity system.

Figure 2 provides an overview of the topics addressed during the unit. The earlier tasks used gear manipulatives as a tool (Engeström 1987, 1999) to support an initial investigation of gear ratios. Later tasks encouraged students to move beyond the physical gears to investigate different ratios (such as a ratio between gears with relatively prime numbers of teeth).

Each of the teacher-researchers/teachers made on-the-spot modifications of the tasks when implementing the unit in order to accommodate the specific nature of their students' developing understanding. These modifications often reflected the teachers' mathematical goals. In this manner, we see how components of the activity system, which include rules (such as a teacher's beliefs), community (such as the students' thinking), and tools (such as the use of gears and the specific tasks) work together to mediate the relationship between the teacher and his or her moves for supporting student reasoning (Fig. 1). All four teachers employed an inquiry-oriented approach in which students solved open-ended problems in groups and individually before discussing ideas together. The instructors followed the students' lead by probing their thinking, asking them to explain and justify their strategies, and encouraging a free exchange of ideas. We videotaped all sessions, took detailed field notes, and collected copies of students' written work. We then transcribed all of the videos, using gender-preserving pseudonyms for all participants.

Teaching experiments

We conducted two teaching experiments (Cobb and Steffe 1983; Steffe and Thompson 2000) taught by two teacher-researchers. The teacher-researchers developed a tentative progression of tasks to foster ratio reasoning through the examination of co-varying quantities. However, the teaching experiment model demands a flexibility that requires any initial task progression to serve only as a rough model for instruction. During and between each session, the teacher-

Mathematical Topic	**Sample Class Activities**
Coordinating relevant quantities	Finding ways to keep track of simultaneous rotations of different-sized gears
Relating teeth to rotations; inverse relationships	Determining how to relate the turns of a gear with 8 teeth to the turns of a gear with 12 teeth
Constructing ratios; constant ratios in non-uniform tables	Finding relationships between gears with 8/12/16 teeth; determining if rotation pairs come from the same gear pair
Connecting $y = ax$ equations to the gear situation	Explaining how $(3/4)m = b$ relates to both rotations and teeth

Fig. 2 Overview of the topics and activities in the linear functions unit

researcher engaged in an iterative cycle of (a) in-the-moment teaching actions, (b) assessment and development of hypothesized models of students' mathematical thinking, and (c) the invention and revision of new tasks in order to test the hypothesized models. In this manner, each teaching session supported a more robust set of hypotheses about the students' understanding based on the previous cycle (Simon et al. 2010).

Ms. A The first teaching experiment occurred over the course of 15 days for 1.5 h each day. The instructor was a professor of mathematics education and the first author. The teaching experiment occurred at an ethnically diverse public middle school (grades 6–8; ages 12–14) in the United States. Seven 7th-grade (age 12) students, 6 girls and 1 boy, participated in the teaching experiment.

Mr. J The second teaching experiment occurred over the course of 13 days for 1.5 h each day. The instructor was a 5th year doctoral student in mathematics education. The teaching experiment occurred as part of a university-sponsored summer program in the United States. The participants were 8 students (5 male and 3 female) who had just completed 8th grade (ages 13–14).

Classroom implementations

Ms. L Ms. L was a 7th and 8th grade mathematics and science teacher with 8 years of teaching experience. She held a bachelor's degree in elementary education with a minor in teaching mathematics. Ms. L implemented the linear functions unit for 10 days, with each lesson lasting 55 min. We collected data in a combined 7th and 8th grade (ages 12–14) classroom in the United States containing 17 students (7 7th-grade students and 10 8th-grade students).

Ms. B Ms. B was an 8th grade mathematics teacher with over 15 years of teaching experience. She was a secondary certified teacher with an undergraduate degree in secondary mathematics education. Ms. B implemented the unit for 11 days, with each lesson lasting 45 min. Her class was a general 8th-grade (ages 13–14) mathematics class in the United States, which contained 22 students.

Data analysis

We followed the constant comparative method (Glaser and Strauss 1967; Strauss 1987; Strauss and Corbin 1990), first transcribing each lesson's video, including gestures, images of written work on the board, as well as student work when possible. Via open coding (Saldaña 2009), we initially focused on three elements of the activity system: Rules (particularly, classroom norms and the teacher's mathematical focus), object (teacher moves), and community (particularly, the students' engagement with the tasks). Five researchers individually coded each lesson, and then compared and discussed the codes as a group, developing an initial list of codes organized around emerging themes. We then examined relationships between codes and solidified a set of themes within each of the elements of the activity system. These themes addressed the different forms of tools, rules, community, and division of labor that mediated the teacher's moves. As we continued analysis of the classroom implementations, we engaged in an iterative process of revising codes and their definitions, as well as identifying new codes. Ultimately this process yielded a stable coding scheme, which we used to support the next phase of focused coding (Saldaña 2009). In the focused-coding phase, two researchers independently re-coded the data set. We continued to discuss and reconcile any differences in coding as a team (Harry et al. 2005), which resulted in further revision and elaboration of the codes. The resulting scheme identified four major functional categories of teacher moves: eliciting, responding to, facilitating, and extending student reasoning.

The teacher moves differed in terms of their potential to support student reasoning. Although some moves offered stronger potential by giving students more responsibility as doers of mathematics, we observed that some moves offered less potential by giving the teacher a more prominent role. These moves often limited students' opportunities to engage with the mathematics in meaningful ways. For instance, consider two responding moves, *correcting student error* and *prompting error correction*. These moves are closely related, but the latter typically provides students more opportunities to reason about a strategy or think through an incorrect idea (Speer 2008). Therefore, we drew both from Speer's discussion about the potential teacher moves can offer to support reasoning and a consideration of how the teachers' moves appeared to affect their students' opportunities to reason (as inferred by their responses to tasks, their written work, and their participation in class discussions) to place teacher moves along a continuum. This continuum addresses the *potential* of each move for supporting student reasoning. We use the term potential deliberately in recognition that each move may differ in its effect on student reasoning depending on a variety of elements in the activity system.

Results

The TMSSR framework

The TMSSR framework groups teacher moves into four categories based on the function they serve in supporting the processes of student reasoning, i.e., searching for similarities or differences, validating, and exemplifying (Jeannotte and Kieran 2017). These four categories are *eliciting*, *responding*, *facilitating*, and *extending* (Fig. 3). As we have noted, teacher moves within each category can differ in their potential for supporting student reasoning. At the same time, we acknowledge that multiple elements in the activity system work together to determine the outcome of any given move, thus the same move can have different outcomes in different circumstances. Therefore, we place teacher moves within each category on a continuum for their *potential* for supporting student reasoning. We use the term potential rather than impact in recognition that our analysis focused more on the classroom discussion than on individual students' performance.

Eliciting Student Reasoning		Responding to Student Reasoning	
Low ⟷ High		Low ⟷ High	
Eliciting Answer	Eliciting Ideas	Correcting Student Error	Prompting Error Correction
Eliciting Facts or Procedures	Eliciting Understanding	Re-voicing	Re-Representing
Asking for Clarification	Pressing for Explanation	Encouraging Student Re-voicing	
Figuring Out Student Reasoning		Validating a Correct Answer	
Checking for Understanding			

	Facilitating Student Reasoning		Extending Student Reasoning	
	Low ⟷ High		Low ⟷ High	
Guiding	Cueing	Providing Guidance	Encouraging Evaluation	Encouraging Reflection
Guiding	Funneling	Encouraging Multiple Solution Strategies	Pressing for Precision	Encouraging Reasoning
Guiding	Topaze Effect	Building	Topaze for Justification	Pressing for Justification
Providing	Providing Information	Providing Alternative Solution Strategies		Pressing for Generalization
Providing	Providing Procedural Explanation	Providing Conceptual Explanation		
Providing	Providing Summary Explanation			

Fig. 3 The TMSSR framework

Within each category, we place moves of a similar form on the same row. For instance, within the eliciting category, teachers can elicit a direct answer to a task or a problem (*eliciting answer*), or they can elicit students' ideas about a task or the concepts relevant to a problem (*eliciting ideas*). Each of the moves are similar, with *eliciting ideas* offering a higher potential for supporting student reasoning. There may also be multiple moves that correspond to one another. For instance, in the facilitating category, the two lower-potential moves, *providing procedural explanation* and *providing summary explanation*, correspond to the higher-potential move of a similar nature, *providing conceptual explanation*.

The four categories themselves are not strictly hierarchical, but they represent a continuum of potential for supporting student reasoning, in that the extending moves were typically, but not always, more effective in fostering the processes of searching for similarity or difference, validating, and exemplifying. Figure 4 represents the relationships between the four categories of moves as evidenced from the teaching-experiment and classroom implementation data, with eliciting at the bottom and extending at the top. Solid arrows represent the "ideal" progression of moves during a classroom discussion, but the participating teachers followed many different patterns of moves, as represented by the dashed arrows.

The moves in the TMSSR framework occurred during all aspects of classroom discussions, including whole-class discussions, one-on-one conversations between teachers and students, and small-group exchanges. Below, we describe each of the four framework categories in turn, and then exemplify aspects of the framework in action by presenting a classroom episode. The episode characterizes the framework in action and identifies the ways in which the instructor relied on particular moves in relation to her broader mathematical goals for the discussion.

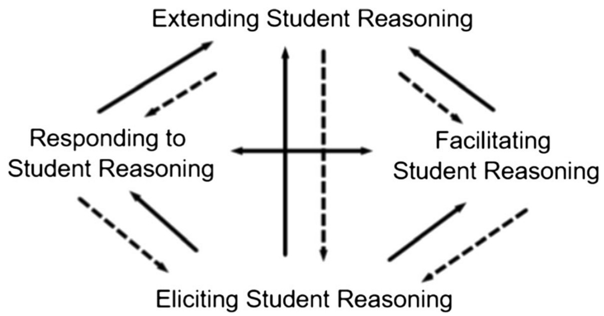

Fig. 4 Relationships between eliciting, responding, facilitating, and extending

Eliciting student reasoning Eliciting moves are those in which teachers aim to draw out, identify, clarify, and understand students' ideas and contributions. Table 1 organizes eliciting moves into low-potential and high-potential moves, providing definitions and data examples of each. Each of the data examples in Table 1 are from one of the four classroom implementations with Ms. L, Ms. B, Ms. A., or Mr. J.

Eliciting moves are important in serving to assist teachers in understanding what students know and understand. They enable teachers to assess students' thinking in the moment, while engaged in discussion with the students. Teachers can elicit basic facts or solution strategies, but they can also elicit students' rationales for their solutions, their understanding of a new idea, the degree to which students' concepts are connected to broader mathematical principles, and their ability to provide coherent explanations for their ideas. When eliciting, teachers are engaged in the act of understanding, assessing, and clarifying students' ideas. These moves can be an important first step in an ongoing process of building on and supporting students' mathematical thinking.

Responding to student reasoning When teachers react in the moment to students' thinking, they can respond in a number of ways: By validating students' responses, by correcting incomplete or inaccurate reasoning or solution strategies, or by encouraging students to take on these roles themselves (Table 2). In some cases, a decision about a code's location depends more on the teacher's inferred intent than on the structure of his or her statement. For instance, one excerpt in Table 2 involved a situation in which Mr. J posed a question to the class. Benito had incorrectly stated that when the big gear makes one rotation, the small gear would make fewer than one rotation. Mr. J asked, "The smaller gear goes less amount, goes around less than once?" From an activity theory perspective, one must consider not only Mr. J's intent, but also the function this move made in the classroom discussion. One could potentially categorize it as an eliciting move, *asking for clarification*. However, Mr. J's intent was to highlight attention to Benito's incorrect reasoning, thus prompting a correction of the error. Therefore, the move is better categorized as *prompting error correction*.

Responding moves commonly followed eliciting. Once students had begun to share their strategies, solutions, and ideas, the instructors then responded in a variety of ways identified in Table 2. The other common set of moves that typically followed eliciting were facilitating moves, which we discuss next.

Table 1 Definitions and examples of eliciting moves

Teacher move	Definition and data examples
Low-potential moves	
Eliciting answer	Asking a question to elicit the answer to a given task. *Mr. J*: "All right, let's start with the easy one. So, if we turned…we turned one of the gears, small gear turns clockwise. What happens to the big gear?"
Eliciting facts or procedures	Soliciting students' recitation of known facts or procedures. *Ms. B*: What's another way we can write another ratio besides using the two dots?
Asking for clarification	Asking a question to clarify the student's meaning. *Leigh*: The middle gear's teeth equals, the middle gear's teeth is 12 and the big gear's teeth are 16. So you need ¾ times 12 and if it equals 16, then… *Ms. L*: So are you wondering this? [Writes "¾(12) = 16?" on the board.]
Figuring out student reasoning	Attempting to understand a student's solution, explanation, or reasoning. *Jorge*: I noticed that the small one goes around more faster than the medium one. *Ms. B*: So that, when you say goes around more faster, what do you mean? *Jorge*: Revolutions. *Ms. B*: So it does more revolutions than the medium one does?
Checking for understanding	Asking a question to assess students' understanding of the mathematical ideas under discussion. *Ms. A*: Would the number of turns the small gear makes and the number of turns the big gear makes ever be the same number? That's a good question, what do you think? *Jill*: No. *Ms. A*: No? Why not? *Larissa*: Because this one has to spin more times in order to make up this one. *Jill*: Yeah, that'll never catch up.
High-potential moves	
Eliciting ideas	Asking a question or questions to elicit students' ideas for a solution strategy or about a mathematical idea. *Ms. B:* Okay, so how can we use this [refers to table written on the board] to help us figure out the number of revolutions that are happening so we know when they match up again?
Eliciting understanding	Assessing what students understand and attempting to identify the nature of students' reasoning. *Tessa*: Two over 1. *Mr. J*: Two over 1, what does that mean? *Tessa*: Every time that one [points to the small gear] does 1 [rotation], this one [the big gear] does 2 [rotations].
Pressing for explanation	Asking students to elaborate on their thinking, explain their reasoning, or share their reasoning. *Hope*: So it's 2.5 and then instead of writing a whole new one I worked by like continuing. So I added 5, even though it isn't proper, and it said 7.5 and then it equaled this one. So, you know, it works because of the 2/3. *Ms. L*: Okay, so can you maybe elaborate a little bit more? Which thing is the 2/3? *Hope*: Okay, so like this is like you're reducing it down a different way. What I did is I would say, like this, you're just making, because, I don't know how to say this. Okay, so like 7 and a half, like, you're trying to find 2/3 of 7.5.

Table 2 Definitions and examples of responding moves

Teacher move	Definition and data examples
Low-potential moves	
Correcting student error	Directly correcting a student error or supplying a more general correct answer. *Ms. L*: So if gear A spins 10 times when Gear B spins 5 times, what's the ratio of spins from A to B? *Ben*: Five. *Ms. L*: There's a difference of 5, right? But we would say 10 over 5, so we would say 2 over 1 if we wanted to reduce it.
Re-voicing	Repeating a student's ideas, either verbally or in written form, to make those ideas public. *Larissa*: The small gear made 2/3 more revolutions than the big gear did, in the same amount of time. *Ms. A*: [Writing this sentence on the board] Small gear made 2/3 more of a revolution than big gear in the same amount of time.
Encouraging student re-voicing	Asking students to re-voice other student ideas or solutions. *Ms. B*: Do you understand what he did? *Students*: Yes. *Ms. B*: Okay. Samuel, can you say in your own words what their group did?
Validating a correct answer	Actively confirming student's idea by re-voicing, re-wording, or adding information to the student's response. *Ms. A*: What's the relationship between the 4 extra teeth and 1/3? *Jill*: Oh, because, because the 4 teeth is a third of the 12 teeth. So when it's spun twice... *Ms. A*: Yeah, that's right. Four teeth is a third of the whole revolution, the whole 12 teeth.
High-potential moves	
Prompting error correction	Prompting students to address their errors. *Mr. J*: If I turn the big gear one rotation, the little gear goes a fraction? Does that mean smaller less amount? *Benito*: Yeah, smaller. *Mr. J*: The smaller gear goes less amount, goes around less than once? *Troy*: More than...the small gear goes around more than once.
Re-representing	Form of re-voicing in which one provides a representation as a way to publicly share a student's idea or strategy. The teacher may organize, re-frame, or formalize the student's statement or work. *Jill*: So like if the, if the small one goes once around, the big one goes 2/3. And if, like going on, it would be ½ and 1, 1 and 1/3, 2 and ½ and 1 and 2/3… *Ms. A*: Okay, so, she has a long table of values with increments of ½ on one side and 2/3 on the other.

Facilitating student reasoning Teachers can respond to students' ideas in more substantive ways by building on their thinking, providing information, explanations, or alternative solution strategies, or encouraging students to develop different solutions. When teachers shift from in-the-moment responses to moves that begin pushing student thinking, we categorize those moves as facilitating, which we address in this section, and extending, which we address in the next section.

Facilitating moves (Table 3) typically occur when a teacher tries to assist students in developing their reasoning through various forms of guidance and explanation. Although the primary responsibility for the mathematics may often remain with the teacher in this category, facilitating moves can help students engage in mathematical

Table 3 Definitions and examples of facilitating moves

Teacher move	Definition and data examples
Guiding moves	
Low-potential moves	
Cueing	Cueing students' attention by indicating that they should focus on a particular aspect of a task, idea, or solution. *Neelam*: So it's hard because for one thing these are prime numbers, and they're not, like, they don't, but, like, if I try to… *Ms. L*: Could you make a table? Like, what would be a good value to put in that table?
Topaze effect	Breaking a task into smaller parts and reducing its complexity by asking easier questions, ultimately revealing the answer through questioning. *Ms. L*: Could these all be from the same gear pair? *Students*: [Silence] *Ms. L*: If it were, what would be true about the ratio of A to B? *Students*: [Silence] *Ms. L*: If they were all from the same gear pair, and we made a fraction with the A value over the B value, what would be true about all of those fractions? *Student*: [Unintelligible]. *Ms. L*: So if Gear A spins 10 times when Gear B spins 5 times, what's the ratio of spins from A to B?
Funneling	Asking leading questions to direct students down a specific path. *Mr. J*: Three and 2 give us 60 teeth, how many teeth in here if we do one rotation of the rear [the back gear], how many teeth is that José? One rotation of the rear would give us how many teeth? *José*: Thirty. *Mr. J*: Thirty, how much for the front if we did 1 and 1/2? What's got to be? *José*: Thirty. *Mr. J*: Okay, now what? How many teeth for a third, a third of a rotation? *Carlton*: Ten. *Mr. J*: How much for a half of the front? *Carlton*: Ten. *José*: Oh, there's a pattern. *Mr. J*: What do we know about the number of teeth if you're going to have similar rotations? *José*: They have to stay the same.
High-potential moves	
Providing guidance	Providing hints, a potential strategy, or another type of conceptual scaffolding without outlining the solution structure. *Mr. J:* When this one, when the small gear turns once, how many times does the medium gear go? *Troy*: Two-thirds. *Mr. J*: Two-thirds, okay. So we want it to go twice as fast and then the other one, number 8, we want to go… *Troy*: Twice as slow. *Mr. J*: Or half as fast or whatever. Half as many turns, okay. So remember what we know. We know right now in this current situation, this goes 2/3 around every time this one goes once. So we want twice as many turns and the other one we want half as many turns. So we're replacing the small gear with a different gear with a different number of teeth so that when it goes around once the big gear will either go twice as many turns or as half as many turns.

Table 3 (continued)

Teacher move	Definition and data examples
Building	Building on students' earlier contributions to support new understanding, or encouraging students to build on one another's contributions. *Ms. A*: You have three ways, number 1 and number 2, number 3. To… you've basically come up with three pairs of numbers for how the gears turn. So my question to you in the next ten, fifteen minutes is, how many other pairs of numbers can you come up with?
Encouraging multiple solution strategies	Encouraging a variety of solution strategies. *Ms. B*: Okay, did a group do it a different way besides breaking up the teeth into groups of 4?
Providing moves	
Low-potential moves	
Providing procedural explanation	Providing a procedural explanation for how to solve a problem by outlining the solution structure. *Ms. B*: We can pick a number and if it works for both our top number and our bottom number, and we get this over here, then they're equivalent. But we can't pick two different numbers.
Providing summary explanation	Summarizing final thoughts about a task or a problem, or providing a summary of information about the task. *Ms. A*: Well, the reason why it's been working is related to what Dorothy was saying, that each of these pairs of numbers is a representation of 2/3. Because it all comes back to the gears. The big gear turns 2/3 as much as the small gear turns every time. So, each of these instances (pointing to the table) of pairs of revolutions, that the big is always going to be 2/3 of the small.
Providing information	Providing general information (rather than providing information specific to a task). *Ms. A*: These are both examples of quantities related to the gears. Quantities are things that you can measure. Like, or things that you can count. So you can measure the circumference of the small gear if you were to lay a string around it as somebody mentioned. Or the teeth of the gear is another quantity, because you can count how many teeth.
High-potential moves	
Providing conceptual explanation	Offering an explanation with a conceptual basis, often focused on explaining why. *Ms. L:* This one has 8 and this one has 12 so this is going to, these 8 teeth are going to meet up. Each of them is going to meet up with a tooth on this big gear. So 8 of these teeth are going to go past the middle part and there are 12 on the gear total…When Gear A goes around once, Gear B is going to go around 8 out of 12.
Providing alternative solution strategies	Initiating a new or different way of solving a problem. *Ms. B*: I'd like to share with you something that someone in seventh hour came up with. So in seventh hour, somebody said, "I'm going to rotate the small gear one – I'm going to rotate the small gear one time. Stop. So my small gear has gone around one time. *Ben*: And the big gear hasn't even, went around a half of a time.

reasoning by encouraging them to make conjectures, identify patterns or compare or classify ideas. They can also involve summarizing students' ideas or introducing conceptually meaningful information into a conversation. Guiding moves are those in which the teacher supports students' reasoning by providing scaffolds. In the case of

Table 4 Definitions and examples of extending moves

Teacher move	Definition and data examples
Low-potential moves	
Pressing for precision	Encouraging students to provide an exact answer, to check their work for accuracy, or to quantify a qualitative statement. *Mr. J*: How do I know when I actually do 60 of these I'm going to get 40 of these? *Caleb*: Because two-thirds. *Mr. J*: Two-thirds what?
Encouraging evaluation	Asking students whether they agree with one another's answers or explanations. *Ms. B*: What do the rest of you think about that? *Ben*: Yes. *Ms. B*: Yeah? Can you show me on your thumbs? Yes I agree, I disagree, I'm not sure about what Ben said?
Topaze for Justification	Initially pushes for justification, but then downgrading the questioning by introducing a series of leading questions in a manner that ultimately reveals the structure of the justification. *Ms. L*: Why does it work? *Lila*: I don't know. *Ms. L*: What did she do with those numbers first? *Lila*: Hmmm. *Ms. L*: Twenty-seven minus 18 is what she did first. So, if we think about this, like basically I think what she started by doing was assuming that it was the correct ratio, okay? *Lila*: Um. *Ms. L*: So, when she took 27 minus 18, if this is three thirds of the, and this is two-thirds, if you subtract those, how many thirds are left? *Lila*: One.
High-potential moves	
Encouraging reasoning	Encouraging students to think about a task conceptually. *Dorothy*: Since you're going up by 3 on that one, you should be going up by 2 because the bigger gear is 2. Oy, the bigger gear is like 4 teeth bigger than what this one is. *Ms. A:* What does the 4 teeth more of the bigger gear have to do with this 3 and 2 here? *Lucy*: Because… *Timothy*: Because that's a third more of the teeth. So it has to go through another third of the teeth.
Encouraging reflection	Asking students to reflect on provided answers or explanations. *Jill*: Working with 1/3 out of the big, the big gear, wouldn't it be the same as taking the half of the small gear? Because it's the same number. Four teeth and 4 teeth. *Ms. A*: What do the rest of you think? She says, isn't 1/3 of the big gear the same as half of the small gear?
Pressing for justification	Asking students to explain why something works or to justify a mathematical idea, strategy, or solution. *Ms. L*: So does that seem correct that if the medium gear spun 12, the big gear would spin 9? *Students*: Yes. *Ms. L*: Okay, you are saying yes. Anyone see a proof of why that works? Something you can use for evidence. Laura?
Pressing for generalization	Encouraging students to generalize their reasoning through formulating a rule, describing a general process, or making connections across cases.

Table 4 (continued)

Teacher move	Definition and data examples
	Ms. A: Okay, can anyone give me a general statement about how to use the number of teeth on each gear to figure out how many times the gears turn? Can you give me a general rule that doesn't depend on 24 as a specific number?

the low-potential moves, these scaffolds may strongly constrain the nature of a classroom conversation. One of these moves is the *Topaze effect*. Similar to the move described as reducing by Stein et al. (1996), we adopt the term Topaze effect from Brousseau's (1997) description of a teacher successively breaking a task into smaller and smaller parts, thereby ultimately revealing the answer through the series of questions. Along with funneling, the Topaze effect move is not one that necessarily supports student reasoning; rather, it can often operate to block opportunities for meaningful conceptual engagement. Despite these limitations, we include these moves in the facilitating category because they often serve as a starting point from which teachers then shift into high-potential moves such as *building* on students' thinking or *providing guidance* as students wrestle with new ideas.

Providing moves are those in which teachers introduce new ideas, facts, procedures, strategies, or conceptual explanations into the classroom discourse. These moves can be conceived as a form of telling (Lobato et al. 2005), with the low-potential moves providing factual information or outlining the solution structure to a task. The high-potential moves, in contrast, introduce mathematical ideas that address conceptual connections, or offer alternate solution strategies after students have shared their solutions.

Extending student reasoning Extending moves foster students' opportunities to extend their mathematical reasoning, particularly in terms of generalizing their strategies or ideas, and developing mathematically appropriate justifications (Table 4). The extending category is on the high end of a continuum for supporting student reasoning precisely because each of the moves reflects an intent to foster more sophisticated mathematical reasoning. Thus, even the low-potential moves can still offer significant opportunities for students to reason about ideas. *Topaze for justification* is an exception in that this move curtails, rather than supports, students' reasoning opportunities. When engaging in Topaze for justification, a teacher's series of questions aimed at eliciting a justification becomes increasingly explicit until the structure of the justification is supplied through the questions. We include this move in the extending category because it represents an intent of the teacher to foster meaningful mathematical reasoning by pressing for a justification.

Classroom episode

The following classroom episode demonstrates how moves within the TMSSR framework operate together in a classroom discussion to support mathematical reasoning. In this episode, Ms. L introduced a task in which students had to determine whether all of the ordered pairs in a table represented rotations from the same gear pair (Fig. 5).

The discussion began with Hope sharing a strategy:

Hope: I made the table again so like, so this is the first one (writes the ordered pair 7 ½ and 5). So, Lewis' formula yesterday was to divide the smaller number, which is the bigger gear, by 2. So, I wrote, so it's 2.5. And then instead of writing a whole new one, I worked by, like, continuing, so I added 5 even though it isn't proper and it said 7.5 and then it equaled this one (points to 7 ½ on the table). So, you know it works, because of the 2/3.

Ms. L: Okay, so can you maybe elaborate a little bit more? [*Eliciting*: *Pressing for explanation*] Which thing is the 2/3? [*Extending*: *Pressing for precision*].

Hope: Okay, so like this (points to the 5) is like you're reducing it down a different way. What I did is I would say, like this you're just making, because – I don't know how to say this. Okay, so like 7 and a half, like, you're trying to find 2/3 of 7.5.

Ms. L: Okay, so that's our whole? [*Eliciting*: *Asking for clarification*].

Hope: What?

Ms. L: The whole is 7 and a half? [*Eliciting*: *Asking for clarification*].

Hope: Yeah, but you're trying to see if it's, if the whole is 7 and a half. So if this (points to 5) is 2/3 of this (points to 7 ½), then you would divide by 2. If this (5) is 2/3 of this (7 ½) you divide by 2 and that, I got 2.5.

Ms. L: (Draws a picture of a circle divided equally into thirds). So, on this picture, don't answer this Hope, on this picture over here, where would that 2.5 go?

The following table contains pairs of rotations for a small gear and a big gear. Did all of these entries come from the same pair of gears, or did some of them come from different gears altogether? How can you tell?

Rotations (Small)	Rotations (Big)
$7\frac{1}{2}$	5
27	18
$4\frac{1}{2}$	3
16	$10\frac{2}{3}$
$\frac{1}{10}$	$\frac{1}{15}$

Fig. 5 The gear pairs table task

[*Facilitating: Providing guidance.*] Okay, talk to your table for 20 seconds and think about where 2.5 is going on here. [*Extending*: *Encouraging reasoning*] Okay, I hear most tables talking. Are you communicating telepathically? Okay, here's our test for our telepathic communicators. Everybody tell me, where does the 2 ½ go? [*Eliciting*: *Eliciting an answer*].

Students: In one of the thirds.

Ms. L: I heard it from some people. It's half of the 2/3, right? Because we're saying this was 5, so we're going 2 ½, 2 ½, 2 ½, right? (Labels each third of the circle with a 2 ½). [*Facilitating*: *Providing a conceptual explanation*] So, if you take that 5 which was 2/3 and you take half of it, this part here (draws arrow around two-thirds of the circle) is the 5, now what's this whole (gestures at the full circle)? [*Extending*: *Encouraging reasoning*].

Students: 7.5.

To determine whether a particular ordered pair, (7.5, 5), represented the gear ratio 2/3, Hope took the smaller number of rotations, 5, halved it to get 2.5, and then added that half back on to 5 to check that it was equal to the larger number, 7.5. Hope's justification, that it works "because of the 2/3", was an ambiguous statement. Ms. L pushed Hope to provide a better justification by *pressing for an explanation*, particularly to clarify what 2/3 represents. Hope responded by describing 5 as 2/3 of 7.5, but not referencing the gear ratio. Ms. L then decided to ask the students to identify the whole; she wanted them to understand that 5 was 2/3 of the whole, and 2.5 was 1/3 of the whole. Ms. L elicited, *pressing for explanation* and *asking for clarification*, to draw out the students' ideas about what constituted the whole. When Hope's response did not clarify the meaning of 2.5 as 1/3 of 7.5, Ms. L shifted to an extending move, *encouraging reasoning*, by asking students to relate 2.5 to one of the circle's thirds. She then shifted back to eliciting before facilitating by *providing a conceptual explanation* with the picture of the circle.

Throughout the episode Ms. L emphasized eliciting, such as asking for clarification, eliciting answers, and pressing for explanation. She did offer conceptual explanations as well, but each time Ms. L offered new information, she quickly followed up with eliciting moves to assess student thinking. One of the rules in the activity system that influenced Ms. L's moves were her mathematical and pedagogical goals, which included (a) clarifying the meaning of ratios as fractions, (b) understanding her students' reasoning, and (c) helping them relate each gear pair to the 2/3 ratio. She therefore relied on extending and facilitating moves in ways that combined with eliciting in order to gauge the degree to which the students made sense of the ideas she introduced.

Ms. L articulated a belief in her students as strong ratio reasoners who had a good understanding of fractions. This belief was apparent in her willingness to push her students to reason about the fractions and their meanings. Ms. L also operated within a set of classroom norms that privileged student understanding and contributions, which is why she returned to eliciting moves after each explanation she offered. Ms. L had shared that she preferred to encourage students to develop their own ideas and solution strategies, resorting to facilitating only after students' ideas were made public. On their own, facilitating moves such as providing guidance or explanations are typically not as powerful as extending moves. However, in combination with repeated and persistent eliciting, the facilitating moves in this episode provided opportunities for students' mathematical reasoning about the relationships between parts and wholes. The combination of eliciting, facilitating, and extending moves in this episode demonstrate how one category does not exist in isolation of the others, and it can be fruitful to make use of a multiplicity of moves in supporting student reasoning.

Discussion

The value of organizing moves on a continuum

The TMSSR framework distinguishes four categories of teacher moves while also distinguishing moves within each category according to their potential for supporting mathematical reasoning. As we discussed in Fig. 5, the organization of the four categories suggests a rough continuum, but the organization does not imply that eliciting moves are undesirable or that extending moves are always the most productive moves. Most inquiry-oriented classrooms will involve teacher moves across all four categories. The distributions of moves is of interest, both across the categories and within each category. The instructors in this study more effectively supported student reasoning when their moves were distributed across the categories, rather than remaining mainly within one or two categories, such as eliciting and responding. This finding mirrors Cengiz et al.'s (2011) study, in which they found that a combination of different types of instructional actions was critical in developing opportunities for extending student thinking. Further, we found that the high potential moves did not always result in a better mathematical discussion or more correct student thinking, but that they instead reflected a more student-oriented discussion. The potential of those "high potential" moves, then, was in their ability to emphasize a focus on the students' ideas, enabling teachers to provide students with a space to engage meaningfully in the processes of mathematical reasoning, including generalizing, conjecturing, comparing and contrasting, developing and inspecting examples, and developing and refining justifications.

Many of the TMSSR teacher moves build on constructs in the literature, but some are emergent. For instance, our eliciting moves have similarity with assessing and diagnosing and assessing understanding (Staples 2007), clarifying questioning and divergent questioning (Driscoll 1999; Frey and Fisher 2010; Hufferd-Ackles et al. 2004), probing student thinking (Herbel-Eisenmann et al. 2013), prompting mathematical reflection questioning (Driscoll 1999), and request for elaboration (Krussel et al. 2004). Our responding moves correspond to moves such as re-voicing or asking students to re-voice (Forman et al. 1998; Herbel-Eisenmann et al. 2013; Lampert and Cobb 2003), guiding by following (Staples 2007), responding to student errors (Lampert et al. 2013), and representing (Lampert et al. 2013; Staples 2007). Facilitating moves have similarities with providing direction or direct explanation and modeling (Frey and Fisher 2010; Krussel et al. 2004), funneling (Bauersfeld 1980; Wood 1998), providing structure or guidance or orienting students (Frey and Fisher 2010; Herbel-

Eisenmann et al. 2013; Krussel et al. 2004; Staples 2007), inviting or requesting participation (Herbel-Eisenmann et al. 2013; Hunter 2008; Krussel et al. 2004; Staples 2007), and demonstrating logic (Staples 2007). Finally, the extending moves are related to existing constructs such as eliciting algebraic thinking questioning (Driscoll 1999), attending to mathematical process and content goals (Hunter 2008; Lampert et al. 2013), and requesting a critique of a method or solution (Evans and Dawson 2017). These connections across studies conducted in different grade levels and educational contexts, including Australasian context (e.g., Hunter 2008; Evans and Dawson 2017), demonstrate that the TMSSR framework is not specific to one context.

There were also a number of moves in the TMSSR framework that are distinct from existing constructs, such as *figuring out student reasoning*, *checking for understanding*, *providing procedural, conceptual, and summary explanations*, *providing alternative solution strategies*, *encouraging evaluation*, *Topaze for justification*, and *encouraging reasoning*. In addition to these emergent teacher moves, the TMSSR framework extends the current literature base by offering a framework of moves specific to mathematical reasoning that (a) goes beyond teacher questioning to include other teacher practices, (b) places moves relative to one another according to their potential for supporting reasoning, and (c) offers a characterization of how clusters of moves can work together to foster meaningful mathematical engagement.

Organizing teacher moves in relation to one another enabled us to attend to how moves from different categories can operate together to either suppress or foster student reasoning. The pedagogical moves teachers make are only one component of instruction, but we have found that they provide a useful lens to understand how teachers support their students' reasoning in the moment. Further, leveraging the activity theory framework enabled us to situate both (a) the moves a teacher makes and (b) the outcome of those moves as mediated by multiple interrelated factors, including the students' backgrounds, ideas, and knowledge, the teacher's goals and beliefs, and the classroom norms. This lens also necessitates the caveat that the framework can only address the *potential* each move has for supporting student reasoning; how a teacher enacts a move and the students' responses to it determines its actual affordance.

The utility of the TMSSR framework

The TMSSR framework could be useful to both teacher educators and teachers. It is not prescriptive; there is no ideal distribution of moves. However, analyzing the concentration of moves across the four categories can support an understanding of the ways in which pre-service and in-service teachers engage with their students in the moment. As suggested by Leach et al. (2014), the framework could also provide a structure or model to support teachers who are learning to leverage more high-potential moves in inquiry-oriented classrooms. For instance, when a teacher frequently elicits answers, facts, or procedures, he or she could be encouraged to elicit ideas and understanding. Or, when a teacher facilitates largely by cuing, funneling, or providing information and procedural explanations, he or she could think about shifts to higher-potential moves such as providing guidance, building, or providing conceptual explanations. Those observing novice teachers in the field may find affordances in using the TMSSR framework to identify the variety of moves teachers implement, and to spot instances in which teachers may be more heavily concentrating their pedagogical attention on one category of moves. In this way, teachers may be assisted in realizing their tendencies to use (or not to use) particular moves and be informed about the broad set of moves that they can use (Towers and Proulx 2013).

Most teacher moves offer a variety of potential affordances. Two teachers could implement the same move and depending on the manner in which it is taken up, one could see markedly different affordances for student reasoning. Further, the same move could be used to promote less sophisticated or more sophisticated mathematical ideas, and the nature of the mathematics being discussed will profoundly affect students' engagement and reasoning opportunities. The TMSSR framework does not account for these differences. Further, teacher moves are only one component of instruction; teachers' beliefs and mathematical knowledge for teaching, the curricula on which they rely, the manner in which they adapt tasks, the use of tools and artifacts, and classroom norms all work together to influence the effectiveness of instruction in supporting meaningful student engagement. The TMSSR framework provides a useful lens on one aspect of this complex interplay of related factors, offering a way to attend to qualitatively different aspects of teachers' engagement. By identifying categories of moves and continuum among them, the TMSSR framework showcases the wide variety of pedagogical acts teachers make in the moment when enacting student-centered, inquiry-oriented lessons.

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