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Initiating and Eliciting in Teaching: A Reformulation of Telling

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We address the telling/not-telling dilemma in mathematics education. Telling is instructionally important, but has been downplayed because of (a) perceived inconsistencies between telling and constructivism, (b) increased awareness of the negative consequences of relying too heavily on telling, and (c) a focus on "non-telling" actions as pedagogical implications of constructivism. In response, we advance a theoretical reformulation of telling as the set of teaching actions that serve the function of stimulating students' mathematical thoughts via the introduction of new ideas into a classroom conversation. We reformulate telling in three ways: (a) in terms of the *function* (which involves attention to the teacher's *intention*, the nature of the teaching *action*, and the students' *interpretations* of the action) rather than the *form* of teachers' communicative acts; (b) in terms of the conceptual rather than procedural content of the new information; and (c) in terms of its relationship to other actions rather than as an isolated action. This reformulation resolves some of the concerns with teaching as telling and helps establish the legitimacy of providing new information within a constructivist perspective on learning.

Key words: Constructivism; Direct instruction; Reform in mathematics education, Teaching practice; Teaching (role, style, methods)

In this article, we develop a case that the pedagogical action of telling has been systematically downplayed in the mathematics education literature. Telling has traditionally meant stating information or demonstrating procedures (Smith, 1996). A perceived inconsistency between telling and constructivism, an understanding of

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the potential negative consequences of relying too heavily on telling, and a focus on the development of a range of "nontelling" actions have all contributed to a de-emphasis of telling. We will demonstrate that there are many reasons why telling can be important in the classroom. Whether or not one should tell is a central tension for teachers and teacher educators concerned with developing a practice connected to constructivist tenets. Although other researchers have recognized the need to expand the set of legitimate telling actions, existing approaches often avoid addressing the direct introduction of new concepts. This, in combination with a lack of conceptual coherence across the set of expanded telling actions, suggests a need to reformulate telling.

Our solution is to offer a reformulation of telling as "initiating" by rethinking telling in three ways: (a) in terms of the *function* rather than the *form* of teachers' communicative acts; (b) in terms of the conceptual rather than the procedural content of the new information; and (c) in terms of its relationship to other actions rather than as an isolated action. By analyzing three instructional episodes from our research data, we develop a variety of initiating actions, demonstrate how initiating is most productively conceived in conjunction with the action of eliciting, and show how initiating can foster students' conceptual growth in mathematics. Through this approach, we intend to legitimize pedagogical actions that may have either lost credibility as researchers and teachers have grappled with the implications of constructivism or that were never thoroughly developed to begin with.

BACKGROUND

Reconciling Telling With Constructivism

Although the constructivist perspective on learning does not dictate pedagogy, it nevertheless implies consequences for teaching (Confrey, 1990; Wood, 1995; Wood, Cobb, & Yackel, 1995). Jaworski (1994), citing and paraphrasing von Glasersfeld (1989), suggested that adopting a constructivist perspective can change teachers' actions and intentions in the following four ways: First, a teacher may begin to differentiate his or her actions that are aimed at generating understanding from those aimed at the repetition of procedures; second, a teacher's focus may shift away from external responses toward what can be inferred about a student's mental actions; third, a teacher may gain understanding that knowledge cannot be directly transferred to students via language; and fourth, given the belief that students are trying to make sense in their experiential worlds, a teacher may become more interested in students' actions as clues to understanding what models students are constructing.

Researchers interested in clarifying implications of constructivism for teaching have identified multiple new teaching actions aimed at helping students develop a conceptual understanding of mathematics. These actions include (a) constructing models of students' understanding of mathematics (Confrey, 1990; Shifter, 2001); (b) choosing and sequencing rich mathematical tasks (Henningsen & Stein, 1997;

Lampert, 1990); (c) predicting student reasoning (Smith, 1996); (d) generating and revising hypothetical learning trajectories (Shifter, 2001; Simon, 1995); and (e) directing classroom discourse (Cobb, Wood, & Yackel, 1993). Teachers are encouraged to listen more carefully to their students, pose interesting problems, and allow students to engage in independent mathematical reasoning and problem-solving activities. These important actions broaden the teacher's role in the classroom.

Traditional telling actions have not been strongly emphasized in these discussions of pedagogy. For example, Hiebert et al. (1997) noted that many of the new instructional actions emphasized in recent literature downplay the teacher's role as a presenter of information: "Periodically, educational reformers have advocated presenting less information, shifting more responsibility to the students to search for or invent the information they need" (p. 36). Researchers interested in reforming teaching actions characterize teachers as listeners, facilitators of appropriate mathematical discourse, and guides through well-chosen problem-solving activities, but rarely as presenters of information (Carpenter, Ansell, & Levi, 2001; McNair, 1998; Shifter, 2001; Stein, Smith, Henningsen, & Silver, 2000). A teacher's ability to tell might be reduced to restating what a student has discovered, or telling the entire class how a student solved a particular problem (Wood & Turner-Vorbeck, 2001).

Why has telling been downplayed? Along with a shift toward constructivist theory and its possible implications for teaching came strong criticism of the transmission model of teaching. The transmission model is based on an assumption that mathematics consists of a fixed set of facts and procedures and that teaching is centered on telling students how to carry out those procedures (Battista, 1994; Rittenhouse, 1998). Teachers' words carry intrinsic meaning and are assumed to be directly comprehended by students (Cobb, 1988). Constructivism challenges the assumptions about mathematics teaching and learning inherent in the transmission model (Confrey, 1990). Researchers maintain that mathematical structures are neither perceived nor intuited, but are constructed by reflectively abstracting from and reorganizing activity (Cobb, 1988; von Glasersfeld, 1995). Thus, when teaching is viewed through constructivist theory, the effective teacher no longer stands in front of the class and imparts facts and procedures to students (Richardson, 2001; Wood, Cobb, & Yackel, 1995). Because telling is a central characteristic of the transmission teaching model, the move away from this model may be associated with a discomfort with *any* type of telling.

Certainly there are many drawbacks of traditional telling actions. The "teaching as telling" practice is undesirable when it (a) minimizes the opportunity to learn about students' ideas, interpretations, images, and mathematical strategies; (b) focuses only on the procedural aspects of mathematics; (c) emphasizes the teacher's authority as the ultimate arbiter of mathematical truth rather than developing the students' responsibility for judgments of mathematical correctness and coherence; (d) minimizes the possibility of cognitive engagement on the part of students; (e) communicates to students that there is only one solution path; and (f) represents premature closure of mathematical exploration.

Beyond these reasons, however, constructivist tenets may also suggest a theoretical justification for emphasizing other teaching actions over traditional telling. The notion that students construct new knowledge based on prior knowledge as they organize their experiential worlds encourages teaching actions centered on listening to students' explanations and attending to their reasoning (Nelson, 2001). The model of students learning through accommodation as they attempt to resolve perturbations implies that teachers should focus on shaping students' progress through carefully chosen questions, problems, and challenges (Simon, 2001). Teachers should generate situations that stimulate children's mathematical activity, and should realize that substantive learning occurs through interaction, conflict, and surprise (Wood, 1995).

Furthermore, Piaget (1970) warned that "each time one prematurely teaches a child something he could have discovered himself the child is kept from inventing it and consequently from understanding it completely" (p. 715). Thus, it is not surprising that research and reform documents frequently develop a range of pedagogical actions focused on listening, guiding, and shaping classroom discourse while downplaying actions centered on introducing new mathematical ideas (Ball, 1993; Lampert, 1991; McClain & Cobb, 1998). For example, the *Professional Standards for Teaching Mathematics* (National Council of Teachers of Mathematics [NCTM], 1991) listed seven roles in discourse for teachers, only one of which addresses providing information. This nod to telling notes that "decisions when to tell students something directly . . . depend on teachers' understanding of mathematics and of their students" (p. 36). The implication is that teachers already know *how* to tell, but need to learn *when* to tell. Telling is folded into a minor support role in the larger system of guiding the development of mathematical discourse: Teachers might occasionally tell students about vocabulary, conversational norms, or facts but must learn when to do so in order to best facilitate the flow of discussion (McClain & Cobb, 1998; Richardson, 2001; Rittenhouse, 1998). We take a different view by proposing that telling actions differ when the goal is to develop concepts as opposed to procedures. These telling actions require further articulation, just as other teaching actions have been more thoroughly developed.

Several researchers have noted the common misconception that constructivism implies a "discovery" view of pedagogy that advises against telling students anything (Clement 1997; Cobb, 1994; Ernest, 1995). The constructivist assertion that "knowledge is the result of a learner's activity rather than of the passive reception of information" (von Glasersfeld, 1991, p. xiv) is sometimes misinterpreted to mean that students *should* construct all knowledge, rather than recognizing that students *do* construct their own knowledge, even in traditional settings. Certainly this misinterpretation is not ubiquitous among all researchers or all teachers. For example, Jaworski (1994) succinctly reminds us that "from a constructivist perspective, students *will* construct for themselves, whatever the teacher does" (p. 137). However, some reports do imply that given teachers' desire to support students' individual construction of knowledge, teachers should not be the "dispensers" of mathematical ideas (Nelson, 2001).

It is not surprising that several studies have suggested that teachers and pre-service teachers frequently interpret constructivist tenets to mean that they should avoid proactive behavior such as telling (Cobb, Wood, & Yackel, 1990; Jaworski, 1994; Philipp, 1995; Rasmussen, Marrongelle, & Keynes, 2003). For example, Jaworski's (1994) ethnographic observations of secondary mathematics teachers included descriptions of teachers who demonstrated a reluctance to tell students facts, even when it became clear their students were not going to discover them on their own. The teachers believed that telling facts would somehow invalidate the students' construction of them. Jaworski described one such teacher and hypothesized that, for her, "it seemed that 'telling' indicated a lapse back into 'transmission teaching', and was therefore something to be avoided." (p. 85).

The question of telling remains a central tension for teachers and teacher educators concerned with developing a practice connected to constructivist tenets. Researchers still grapple with questions concerning the role of formal knowledge (i.e., conventionally accepted disciplinary knowledge), specifically how formal knowledge should be introduced, and at what point during instruction it is appropriate to tell students formal knowledge (Richardson, 2001).

A Rationale for Telling

Concerns with never telling. There are many reasons to be concerned with a practice of never telling. Chazan and Ball (1999) argued that an exhortation to avoid telling can be disempowering to teachers, preventing them from playing a strong role in the classroom. Following this suggestion could reduce teachers' sense of efficacy (Smith, 1996). Furthermore, urging teachers not to tell focuses only on what not to do, leaving teachers with an inadequate model for how to move students forward during times when student-student interactions fail to generate the ideas necessary for mathematical growth. Clarke (2001) points out that the practice of structuring classrooms around student-student interactions, while potentially valuable, does not guarantee that the interactions will be purposeful and effective. Similarly, whole class discussions in which students take turns sharing their solution strategies will not necessarily generate learning. Romagnano (1994) describes these concerns as a telling/not-telling dilemma that has emerged for teachers—telling can pose the danger of restricting further mathematical exploration, but never telling can result in students disengaging with the mathematics or engaging at a superficial level.

Why tell? Sometimes new ideas need to be introduced in the classroom. Hiebert et al. (1997) note that:

> The hands-off approach is overly conservative. It underestimates students' ability to make sense of powerful ideas and ways of thinking that teachers can share with them. In addition to respecting students as thinkers, teachers must respect mathematics as a discipline. (p. 30)

Introducing new information into the classroom can serve as a catalyst for developing new ideas. Although it is unlikely that students will interpret new information exactly as teachers anticipate, it can be helpful to state facts, share ideas, or identify conflicts, and then examine the sense that students make of them. In addition, introducing new information at critical junctures could help reduce the number of problem features that students must attend to, thus allowing for exploration in new areas.

The idea of telling in a way that respects the constructed nature of knowledge has a long tradition, reaching back to Dewey. He noted that teachers should provide information if it is required for students to continue their problem-solving efforts and if they cannot readily find it themselves (Dewey, 1933). If facts and ideas are presented as something to consider and not as a prescription to follow, teachers can tell without fear of suppressing students' mathematics. Furthermore, telling is sometimes necessary for practical reasons. Students cannot be expected to reinvent entire bodies of mathematics, regardless of how well each concept is problematized by well-chosen tasks (Clarke, 1994; Romagnano, 1994). Teachers are expected to enculturate students into the mathematics community, sharing conventional norms associated with mathematical discourse, representation, and forms of argument (Becker & Varela, 1995; Cobb & Yackel, 1996; Driver, 1995). If teachers are to facilitate this enculturation, then making the ideas and conventions of the community available to students is essential. From this perspective, some information must be introduced by the teacher. In short, a telling/not-telling dilemma has emerged. Telling is instructionally important, but has been downplayed due to both perceived inconsistencies with constructivism and historical attempts to develop pedagogical implications of constructivism. Fortunately, this dilemma has not gone unnoticed.

Responses to the Telling Dilemma

Telling offloaded onto students. Rather than telling or showing a new mathematical idea, teachers may relegate telling actions to their students by asking them for the idea or by waiting until more knowledgeable students articulate the idea (Cobb, Wood, & Yackel, 1993). For example, in some of the vignettes presented in the *Professional Standards for Teaching Mathematics* (NCTM, 1991), potential standstills are avoided when one or two students suddenly generate ideas that move their classmates forward (e.g., see pp. 42–44). This pedagogical approach seems dependent on the presence of students who either already have the targeted knowledge or who are at the point of formulating the idea asked for by the teacher. Of course, many students in a given class may not yet be at that point. Furthermore, students might not tell in a way that is productive for the rest of the class or in a way that moves the class forward toward any helpful mathematical goals. The greater conceptual and pedagogical knowledge that teachers bring to the discussion can make a positive difference in terms of presenting new information in ways that enable students to work with those ideas.

Judicious telling. In a second approach, teachers act primarily as facilitators and resort to telling only occasionally. Judicious telling allows teachers to remain focused on students' mathematics while occasionally introducing new information, such as conventional notation or terminology, different representations, counterexamples to students' ideas, or the articulation of a student's solution process (Ball & Chazan, 1994; Driver, 1995; Smith, 1996). In the judicious telling approach, telling actions are particularly encouraged in terms of teaching critical vocabulary, rules, and conventional norms in mathematics when absolutely necessary (Rittenhouse, 1998). An important feature of the judicious telling model is that the teacher is no longer viewed as the sole source of knowledge, but still has some freedom to introduce new knowledge into the classroom.

Expanded telling actions. Other researchers have begun to address the telling dilemma by rethinking what it means to tell. For example, Chazan and Ball (1999) suggested a middle ground between telling and not telling by expanding the definition of telling to include the following teaching actions: (a) introducing conventional terminology, (b) reminding students of a conclusion on which they have already agreed, (c) rephrasing students' comments for the whole class, (d) telling students that an idea is unclear, (e) challenging students' ideas when they reach a mathematically incorrect consensus, (f) managing noncontent related behavior, (g) changing the focus of a discussion, and (h) "inserting a new voice" via questions and comments. In a similar way, Hiebert et al. (1997) expanded the definition of telling to include sharing mathematical conventions, suggesting alternative solution methods, introducing more clear or efficient recording techniques, and articulating ideas in students' solution methods. They argued that telling is legitimate if it does not take fundamental agency for making sense away from students:

> We agree with Dewey that the teacher should feel free, and obligated, to share relevant information. Too much information is being shared only if it is interfering with opportunities for students to problematize mathematics. In other words, information can and should be shared as long as it does not solve the problem, does not take away the need for students to reflect on the situation and develop solution methods that they understand. (p. 36)

Why Reformulate Telling?

Rather than simply including traditional telling in an expanded set of teaching actions, as indicated by the judicious telling approach, we argue instead for a rethinking of the conceptual roots of telling. Specifically, we reformulate telling in three ways: (a) in terms of the *function* rather than the *form* of teachers' communicative acts, (b) in terms of the conceptual rather than the procedural content of the new information, and (c) in terms of its relationship to other actions rather than as an isolated action. By reformulating telling across these dimensions, we also extend the expanded telling approach by legitimizing the pedagogical action of directly conveying concepts and ideas.

Form versus function. In the judicious telling approach, telling is defined in terms of the *form* of the teacher's utterance; traditional telling occurs when teachers make *declarative statements*. However, it is possible for teachers to use a series of questions in such a way that the questions actually tell. For example, teachers can "tell" their students how to perform procedures by "funneling" students toward the correct answer via a tightly guided sequence of specific questions (Bauersfeld, 1980; Wood, Cobb, & Yackel, 1995). In contrast, declarative statements may sometimes question more than they tell. Teachers can make statements that highlight contradictions in students' ideas, provoke new ways of thinking about ideas that lead to new questions, or otherwise encourage student explanations. Thus, defining telling in terms of form alone does not account for the times when questions tell and declarative statements question.

In our effort to find an alternative basis for defining telling, we looked for a common characteristic across the dozen or so pedagogical actions identified by other researchers as expanded telling actions (Chazan & Ball, 1999; Hiebert et al., 1997). In each case, the teacher seemed to take an assertive or proactive action, and in many cases, the teacher made a substantive contribution that influenced the direction of the conversation. However, the actions fulfilled a range of pedagogical functions, from developing ideas and strategies that originated with students to managing general classroom behavior.

A few actions seemed to fulfill a function closely related to telling, namely, that of introducing new information. These actions include introducing conventional terminology, sharing new and alternative solution methods, and suggesting different ways of recording information. The action of "inserting a new voice" (Chazan & Ball, 1999) might also fit the function of introducing new information into the classroom discussion. The authors describe an episode in which a teacher reminded students that they had used a different approach in a previous lesson, thus introducing information that did not exist in the conversation on that particular day. However, the authors do not discuss the direct communication of new concepts that originate with the teacher, perhaps because it would appear to be too similar to traditional telling. For example, Chazan and Ball (1999) provide an example in which a classroom discussion faltered as students argued over the meaning of an average. Certainly just telling students how to compute an average would likely have shut down the discussion without contributing much to their understanding of the concepts related to averages. We would nevertheless like to examine the legitimacy of providing new information related to the meaning of concepts such as average. Later we elaborate a reformulation of telling in terms of the function of introducing new information, in part to validate the type of telling that has historically been most difficult to reconcile with constructivism.

Procedural versus conceptual content. In the judicious telling approach, telling is defined in terms of procedural content. Smith (1996) points out that the practice of "teaching mathematics by telling" is grounded in a mutually reinforcing system of beliefs about mathematics, teaching, learning, and mathematical authority:

> The plausibility of telling depends on the underlying conception that mathematics is a fixed and finite collection of procedures for computing answers. Because students do not know these procedures, they must derive their knowledge from teachers. As a result of teachers' reliance on step-by-step procedural display, students are required to listen carefully and practice diligently. (p. 391)

We argue that telling must be reconceived if a teacher is to tell something with conceptual rather than procedural content. By conceptual content, we refer to ideas, images, meaning, why a procedure works, one's comprehension of a mathematical situation, and connections among ideas.

The two teaching episodes presented by Thompson, Philipp, Thompson, and Boyd (1994) illustrate the difference between telling procedural versus conceptual content. Each of the two teachers opened his lesson with the same mathematical problem:

> At some time in the future John will be 38 years old. At that time he will be 3 times as old as Sally. Sally is now 7 years old. How old is John now?

After the students had worked on the problem and discussed it with a partner, each teacher conducted an interactive discussion. At one point during the discussion, Teacher 1 told procedural content by reviewing the long-division algorithm. In contrast, Teacher 2 told conceptual content by providing information regarding the relationship between Sally's age and John's age (specifically that the multiplicative comparison or ratio of their ages does not remain constant over time). Although Teacher 1 spoke in terms of numbers and arithmetic operations, Teacher 2 spoke in terms of measurable attributes of objects in the situation and of relationships among quantities, contrasting the invariant additive comparison of ages over time with the varying multiplicative relationship between ages over time. It is not difficult to imagine differences in the nature of the ideas that students could construct from these two types of information.

If telling actions differ when the teacher intends to develop concepts, then these actions will require further articulation, just as pedagogical acts related to listening and leading classroom discourse have been differentiated into a range of actions, each being more fully developed. Instead of focusing only on *when* to tell, we must also address *how* to tell, *wha*t to tell, and *why*.

Isolated action versus interrelated actions. From a constructivist perspective, "all learning involves the interpretation of phenomena, situations, and events, including classroom instruction, through the perspective of the learner's existing knowledge" (Smith, diSessa, & Roschelle, 1993, p. 116). As a result, we argue that conveying new information is profitably followed by a pedagogical action intended to ascertain how students interpret the information. Furthermore, decisions to tell are not made in isolation from students, but rather unfold in response to the dynamics of the classroom. Telling is often preceded by the teacher gathering information about students' thinking before making a judgment about whether to shape students' ideas further or to introduce new information. However, once the teacher engages in telling, he or she then steps back to assess what sense the students have made of the new information. By considering telling as part of a system of actions, we focus attention on the development of the students' mathematics rather than on the communication of the teacher's mathematics.

A REFORMULATION OF TELLING AS INITIATING

Definition of Initiating

We define *initiating* as the set of teaching actions that serve the *function* of stimulating students' mathematical constructions via the introduction of new mathematical ideas into a classroom conversation. Redefining telling in terms of the function of the teacher's utterance in the classroom addresses the limitations that we previously identified with defining telling in terms of the form of the communicative act. Three related features influence the function of an utterance: the teacher's *intention*, the nature of the teaching *action*, and the students' *interpretations* of the action. By defining initiating in terms of its function in the classroom, one must accept some degree of uncertainty—it is not possible to ascertain the function of initiating actions with complete conviction. However, it is frequently possible to reasonably infer the likely function an initiating action plays, as we will demonstrate below. Additionally, we view this uncertainty as a necessary trade-off for reformulating telling in terms of its function rather than its form.

Intention. With traditional telling, the intention is typically that students will reproduce a procedure or a definition. Confrey (1990) stated, "When one applies constructivism to the issue of teaching, one must reject the assumption that one can simply pass on information to a set of learners and expect that understanding will result" (p. 109). Thus, with initiating, the goal is not the internalization and/or verbal reproduction of a fixed and unitary concept. Instead, when a teacher introduces a new idea into the conversation, he or she recognizes that the idea will be interpreted in multiple ways. Furthermore, when a teacher engages in initiation, his or her general intention is to prompt coherence and sense making, rather than to induce the reproduction of a procedure or a definition. Because the focus of initiating is on conceptual rather than procedural content, the teacher introduces ideas, images, connections, and underlying meaning for mathematical symbols, rather than procedures. Teachers may have a range of specific intentions for different initiating acts. For example, a teacher may intend to create a space in which students can engage with and further develop the new ideas that are introduced. As a second example, a teacher may intend to provoke disequilibrium in students' thinking by providing new information or a counterexample, thus opening the possibility for the reorganization of students' schemas.

Action. Teachers' intentions can be instantiated pedagogically in a variety of ways. Initiating includes, but is not restricted to, the following actions:

1. Describing a new concept (which can include an idea, the meaning associated with a mathematical symbol, why something works, an image, a relationship, or connections among ideas or representations)
2. Summarizing student work in a manner that inserts new information into the conversation
3. Providing information that students need in order to test their ideas or generate a counterexample

4. Asking students what they think of a new strategy or idea (perhaps from a "hypothetical" student)
5. Presenting a counterexample that the teacher has not seen any students introduce and thinks no one will
6. Engaging in Socratic questioning in an effort to introduce a new concept
7. Presenting a new representation

Any given action can take many forms and can be declarative or interrogative in nature. Furthermore, interrogatives can range from targeted funneling to posing a more open-ended question. The list of initiating actions above is not exhaustive. We illustrate a range of actions later in this article by including empirical episodes of Actions 1–3. By describing these actions in context, we hope to demonstrate that they can occur in settings that are compatible with a constructivist philosophy and the goals of the reform movement.

Interpretation. We believe that intention and action are insufficient to define the function of initiating. One could introduce new information that is beyond students' ability to understand, in which case the action would serve little productive function in the classroom. Students must engage with the new information in some way in order for an action to count as initiating. Thus, students should be conceptually able and motivated to make sense of the teacher's utterance. Engagement is not as stringent a requirement as effectiveness. We focus on engagement to ensure that the teacher does not talk past the students. We maintain that one should not think of an initiating act as one that concludes with the end of the teacher's utterance. Because the purpose of initiating is to stimulate novel mathematical thoughts for students, one must consider students' responses to the teacher's initiating action.

Situating Initiating in Conjunction with Eliciting

One typically conceives of traditional telling as an isolated act and defines it in terms of what the teacher does. In contrast, we conceive of initiating in terms of students' development, even though the new ideas originate with the teacher rather than with the students. Thus, we consider initiating actions not in isolation, but in conjunction with another teacher action. In particular, initiating is often followed by *eliciting*—an action intended to ascertain how students interpret the information introduced by the teacher. Continuing to define pedagogical actions in terms of function rather than form, we consider eliciting to be the set of teaching actions that serve the function of drawing out students' mathematical ideas.

Examples of eliciting. Eliciting occurs when the teacher's actions serve the function of drawing out students' images, ideas, strategies, conjectures, conceptions, and ways of viewing mathematical situations. For the purposes of this article, we do not include the action of determining whether students can recite steps in a procedure as eliciting, although that may indeed be an important pedagogical action at times. Instead we restrict eliciting to the set of actions aimed at identifying students' concepts. We adhere to this restriction because we only intend to examine those pedagogical actions concerned with the generation of conceptual growth rather than procedural competence.

The intention to elicit might be to gain insight into what makes a problem difficult for students or to determine their understanding of a particular idea. The action of eliciting cannot be conceived only in terms of asking a question, since questions may at times serve a telling role. Instead, eliciting actions occur when the teacher arranges for situations in which students articulate, share, discuss, justify, reflect upon, and refine their understanding of the mathematics. The teacher may elicit by posing a carefully designed task or by asking one student to react to the ideas of another student. In some classrooms, the social contract between the teacher and the students might result in students interpreting the teacher's eliciting act as a probe for recitation or mimicry of the teacher's ideas. In this case, the teacher's attempt would have failed to function as a genuine elicitation. Students must respond with their own ideas for an action to count as eliciting.

The relationship between initiating and eliciting. In order to promote students' conceptual development, we believe initiating is most profitably used in conjunction with eliciting. While the agenda framing this development is initially the teacher's, it is iteratively modified in response to the progress of the discussion in order to accommodate students' prior and emerging understanding. Initiating and eliciting acts introduce ideas from different sources. If a mathematical idea originates with the teacher, he or she is operating in an initiating mode. If an idea originates with the student, then the teacher is likely operating as an elicitor.

Because many actions serve dual purposes, it is not always possible to determine whether an action is best described as initiating or eliciting. This uncertainty is compounded by the fact that intentions can be difficult to infer. In addition, the two categories of initiating and eliciting are not intended to include all teaching actions. However, our purpose is neither to describe all possible teaching actions nor to develop a research coding tool to categorize them. Rather, our goals are to reformulate telling, elaborate and develop several initiating actions in context, and situate telling within a framework of closely related categories of teaching actions. Initiating and eliciting interact together; they are not mutually exclusive categories. Rather than describing a simple dichotomy such as "tell or not tell," we show how both categories of actions mutually influence one another as they occur in concert to foster students' conceptual growth in mathematics.

How Initiating Can Support Learning

We claim that initiating can facilitate students' conceptual growth. From a constructivist perspective, learning is viewed as conceptual transformation or restructuring:

> The learning theory that emerges from Piaget's work can be summarized by saying that cognitive change and learning in a specific direction take place when a scheme,

> instead of producing the expected result, leads to perturbation, and perturbation, in turn, to an accommodation that maintains or reestablishes equilibrium. (von Glasersfeld, 1995, p. 68)

Because learning can be triggered by a disequilibrating experience for the learner, one can conceive of an associated teaching action of creating provocations that could be perturbatory for students, leading them to make accommodations in their knowledge. Initiating in the form of asking students what they think of a new idea from a "hypothetical" student or presenting a heretofore unmentioned counterexample could create disequilibrium for students. According to Steffe and D'Ambrosio (1995), creating such provocations requires a deep appreciation by teachers of their students' mathematical knowledge.

Later in his research program, Piaget added a second mechanism of equilibration—reflective abstraction[1]—in an effort to explain how we come to "know about our own processes of knowledge, or about the coordinations of our own actions" (Campbell, 2001, p. 4). Our analysis of the empirical episodes presented later in this article suggests that the initiating actions from these episodes may be more closely related to this second learning mechanism. Reflective abstraction refers to the processes of creating mental records of experience, comparing those records, and identifying commonalties among the mental records related to the learner's goals (von Glasersfeld, 1995).

Arguing that reflective abstraction is underspecified for guiding pedagogical actions in mathematics teaching, Simon, Tzur, Heinz, and Kinzel (2004) articulate a mechanism that they call "reflection on activity-effect relationships" as an elaboration of reflective abstraction. First, the learner attends to the effects of his activity relative to his goal. Then he creates mental records of this experience. The unit of experience being recorded is an enactment of his activity associated with its effect. The learner sorts the records in relation to his goal and compares those records that were grouped together, identifying relationships among the activity and effects. Eventually, the relationship can be treated as a mental action on which the learner can operate. Mathematics is initially built from activities such as counting, folding, ordering, and comparing (Confrey, 1990). Then the learner progresses through different levels of abstraction of his or her actions, including (a) isolating it in the experiential flow and grasping it as a unit, (b) registering it in working memory so that it can be re-presented in absence of "concrete" materials, and (c) taking the result of actions as a stand-in for the action itself (Steffe & Cobb, 1988).

We hypothesize that initiating can support reflective abstraction in at least two ways. First, when students have reached the stage at which they are able to treat constructed regularities as conceptual objects, they may be able to listen actively to new information and use it as a mental object upon which to operate, reflect, and

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[1] The French term "*abstraction réfléchissante,*" used by Piaget, is translated as "reflective abstraction," "reflecting abstraction," "reflected abstraction," and "reflexive abstraction." We use the term *reflective abstraction*, because it is the term most frequently used by mathematics educators.

coordinate with other mental objects. At this point, it may be inefficient and unnecessary to have students remain focused on concrete activity (Thompson, 2000). Furthermore, presenting conceptual content may in fact support the higher level of reflective abstraction at which the student is operating.

Whether initiating is appropriate or fruitful depends upon whether students have sufficient prior experiences with which to make sense of the new ideas. Schwartz and Bransford (1998) conducted a study in which college students engaged in "readiness activities," which consisted of analyzing and contrasting simplified experimental designs with data from classic psychology experiments. Some of the students then attended a lecture addressing the psychological principles related to the phenomena highlighted in the experiments. The students who had engaged in the readiness activities and attended the lecture performed significantly better on a task related to the lecture content than both the students who only engaged in the readiness activities and the students who only attended the lecture. According to Schwartz and Bransford, the teacher's speech was meaningful for the students who had participated in the readiness activities because they could interpret it as relevant to knowledge that they had already developed. Alternatively, when telling occurs without readiness, "the primary recourse for students is to treat the new information as ends to be memorized rather than as tools to help them perceive and think" (p. 477). Thus, pedagogical decisions about initiating should be informed by what Simon (1995) refers to as the teacher's hypotheses about students' knowledge and learning trajectories.

We hypothesize that initiating actions may also support the aspect of reflective abstraction that involves focusing on and isolating mathematical properties or regularities from the experiential flow. For example, a teacher may relocate part of a very complex task onto herself by providing new information, in order to enable students to focus on a smaller part of the phenomenon and engage in mental activity with that component. This might serve to usefully reduce the number of features a student must attend to within a given problem, thus allowing students to productively explore certain aspects of a problem without becoming overwhelmed. As an alternative, a teacher might introduce a new idea into the conversation in order to help set a conceptual goal and focus for the students' subsequent explorations, in essence creating a conceptual space in which students develop new ideas. According to Lobato, Ellis, and Muñoz (2003), these initiating actions can be seen as *focusing phenomena,* which refer to the multiple agents in the instructional environment that contribute to the activity of directing students' attention toward particular aspects of mathematical activity and away from others. We will further describe the ways in which initiating could support processes of reflective abstraction through the three data episodes below.

THE EPISODES AND TEACHING EXPERIMENTS

The three empirical episodes that will be analyzed in the next section exemplify aspects of initiating in conjunction with eliciting. The episodes illustrate Initiating Actions 1–3, respectively, and demonstrate a variety of patterns with initiating and eliciting. Episode 1 (the meaning of division in a rate situation) portrays a cyclic pattern of initiating and eliciting. Episode 2 (the comprehension of steepness in a wheelchair ramp situation) portrays an extensive series of eliciting acts followed by an initiating act. Episode 3 (the creation of ratio as a measure of steepness of a ramp) portrays a pattern of eliciting followed by initiating and eliciting occurring together. The episodes described below are not short excerpts of a transcript illustrating one action. Instead, they are presented in narrative form in order to address all aspects of the function triad—intention, action, and interpretation. Each episode is described in detail in order to develop different ways in which initiating and eliciting can interact in the classroom. This way we can illustrate how the pedagogical actions are used to support students' conceptual development.

The episodes were selected from two different teaching experiments, a summer teaching experiment and an after-school teaching experiment. In both cases, the first author was the teacher. We chose to rely on teaching-experiment data rather than regular classroom data primarily because we can make legitimate claims about the teacher's intention. Specifically, we were able to rely on detailed field notes regarding the goals for each lesson when considering intention.

The summer teaching experiment included 9 high school students split evenly across Grades 8, 9, and 10. We recruited average-performing students earning Bs or Cs in their regular mathematics classroom. The sessions consisted of 30 hours of instruction over 2 weeks, conducted in a university computer lab (for details of the study, see Lobato & Thanheiser, 2000, 2002). The after-school teaching experiment included 8 students, earning grades A–C, recruited from a ninth-grade Algebra I class. It consisted of hour-long sessions held weekly after school for a total of 30 lessons, in which students met in pairs. The major mathematical goal of both teaching experiments was to help students develop a robust and generalizable understanding of rates of change.

We had several goals in performing the analysis of the teaching episodes. First, we intended to illustrate a range of initiating actions. However, we soon discovered that the relationship between the data analysis and our conceptual framework was reflexive. Specifically, our analysis informed revisions of the conceptual framework as well as the particular list of initiating actions. By conducting this analysis, we were able to establish the legitimacy and value of reformulating telling in terms of function. The analysis helped us examine the ways in which initiation can support conceptual development, and it demonstrated the value of conceiving of initiating in relationship to eliciting rather than as an isolated action.

In our analysis of the teaching episodes, we elaborate a variety of patterns of interaction between eliciting and initiating. These patterns differ from other well-known discourse patterns. For example, Mehan (1979) identified a dominant classroom discourse structure of "Initiation-Reply-Evaluation" (I-R-E). In this structure, the teacher initiates by asking a question about a known fact or idea, students reply with answers, and the teacher evaluates the responses for correctness. The I-R-E pattern suggests that teachers often guide students to correct responses by evaluating their answers. Bowers and Nickerson (2001) found two other interaction patterns through an analysis of a course for prospective mathematics teachers. In the "Elicitation-Response-Elaboration" (E-R-E) pattern, teachers elicit a response, students respond, and teachers elaborate on the response. The instructor's elaboration suggests an attempt to encourage deeper conversations. In the "Proposition-Discussion" (P-D) pattern, a student or a teacher makes a proposition, and then other class members discuss it.

Our use of the term "initiating" differs from these existing uses. In Mehan's I-R-E pattern and in Bowers and Nickerson's E-R-E pattern, a teacher initiates a question that has a known answer. Neither our "initiating" nor "eliciting" acts address these types of teacher questions; our use of "initiating" is limited to when teachers introduce new mathematical ideas into a conversation. Although this may occur in the I-R-E or E-R-E patterns, the terms do not address a teacher's deliberate attempt to convey new information. Mehan also distinguished between "product elicitations," in which a series of questions are designed to elicit correct, factual responses, and "process elicitations," a series of questions designed to elicit learners' views and opinions. The latter is similar to our use of "eliciting," in which teachers engage in actions that draw out students' mathematical ideas. In contrast to Mehan's discourse structures, however, our initiating-eliciting framework serves as a way to address specific teaching actions in terms of their function in the classroom when teachers aim to promote conceptual development.

ANALYSIS OF THREE INSTRUCTIONAL EPISODES

Episode 1: Initiating by Describing a New Concept

We present this episode with three goals in mind. First, we illustrate the initiating action of describing a new concept (Action 1 from the list above) in order to explore the legitimacy of directly conveying concepts and ideas within a constructivist perspective. Second, we demonstrate our proposition that initiating is more productively conceived as an action to be performed in conjunction with eliciting. This particular episode demonstrates a pattern of eliciting both preceding and following initiating (an E-I-E pattern). Third, we demonstrate how initiating and eliciting actions are informed by the teacher's hypotheses about the student's knowledge and trajectory for conceptual growth.

Background: The teacher's understanding of the student's understanding. This episode is drawn from Session 9 in the after-school teaching experiment in which the first author tried to facilitate the understanding of division in rate situations for a ninth grader, Carissa.[2] Prior to the teaching experiment, we assessed the participant's understanding by using the following task in a semistructured interview:

[2]Carissa is a gender-preserving pseudonym, as are all other participants' names reported in this article.

> Suppose you collected 16 ounces of water over a period of 24 minutes from a leaky faucet. How fast is the faucet leaking?[3]

One way to understand this situation conceptually is to mentally join the values of the two quantities into a composed "16 oz in 24 min" unit (Lamon, 1995). By iterating (replicating the unit to produce another composed unit) or partitioning (separating the unit into a specified number of equal parts), we obtain values that represent the same dripping rate (e.g., 4 oz in 6 min). To determine the number of ounces that drip in 1 min, we mentally partition the 16:24 unit into 24 equal parts, which involves partitioning both the 16 oz and the 24 min into 24 sections. Determining the amount in each section can be accomplished through division by 24. The concept of connecting partitioning with the arithmetic operation of division is the mathematical focus of Episode 1.

Carissa did not appear to understand these ideas prior to the teaching experiment. In response to the leaky faucet task, Carissa calculated $24 \div 16$ rather than $16 \div 24$ "because it was higher, the 24 min," which suggests a variation of the "division makes smaller" conception (Greer, 1992). She did not interpret 1.5 correctly as minutes per ounce but rather explained that "it's dripping 1.5 oz" and that she was not sure how much time it took "to do the 1.5 oz." This suggests that she used the measurement units as a label rather than linking them to the mental operation of partitioning. It also suggests a lack of understanding of the effect of dividing. Carissa chose the operation of division through a process of elimination based upon number size (as reported in Sowder, 1995), rather than based on an understanding of the situation.

In the eight sessions of the teaching experiment leading up to Episode 1, Carissa explored speed situations using the simulation software MathWorlds (Roschelle & Kaput, 1996), developed for the SimCalc Project. During this time, Carissa (a) formed composed units of distance and time such as "10 cm in 4 s," (b) created her own line segment representations of composed units, (c) iterated composed units to create other "same speed" values, (d) linked iteration to multiplication, (e) demonstrated an understanding of the partitive meaning for division by correctly creating an appropriate word problem for a given division problem, and (f) created unit ratios by dividing time and distance quantities by the same amount. For example, in Session 8, the teacher asked Carissa to find how far a duck (one of the characters in MathWorlds) traveled in 1 s given that the duck covered 7 cm in 3 s. Carissa calculated $7 \div 3$ and $3 \div 3$ to determine that the duck traveled 2.33 cm in 1 s. She appeared to understand that dividing produces the distance traveled per second. However, it was less clear whether Carissa had connected dividing by

[3]We posed an ambiguous question because we were interested in assessing students' conceptions of measures of rates. We would have considered an answer of 16 oz in 24 min, with an appropriate explanation, as correct. However, all of the interview participants interpreted this question as a prompt for a unit ratio. We also used the leaky faucet context for preceding tasks and communicated that the faucet was dripping at a constant rate.

3 with partitioning 7 cm and 3 s into three equal parts. Consequently, in Episode 1 (Session 9 of the teaching experiment), the teacher tried to assess Carissa's understanding of the connection between dividing and partitioning.

Eliciting. In order to elicit Carissa's meaning of division, the teacher drew a line diagram (based on those that Carissa developed) to represent the duck traveling 7 cm in 3 s (Figure 1) and asked, "To find how far he walked in 1 s, you divided the 3 s by 3 and the 7 cm by 3; so can you *show* the dividing by 3 in your picture?" Carissa appeared to understand the question but responded by partitioning the segment into seven parts rather than three parts. When the teacher asked Carissa why she split the segment into seven parts, Carissa responded, "Because I was gonna show where they divided where . . . like, um, like right here that's one [pointing to the first tick mark as shown in Figure 2] and the 2.33 will go here [pointing between the second and third tick marks as shown in Figure 2]." The teacher suspected that Carissa was attempting to mark her "answer" of 2.33 cm in 1 s on the number line but could not do so unless she first marked each centimeter on the segment, as she had seen on the computer screen. This might explain why she partitioned the segment into seven equal parts. The teacher also concluded that Carissa had not associated division by 3 with partitioning the segment into three equal parts and that Carissa was applying a counting scheme rather than a partitioning scheme.

Initiating. To address this issue, the teacher initiated by describing the concept that dividing by 3 is connected with partitioning the 7 cm in 3 s segment into three equal parts:

Teacher: Now take a look at this [draws a new 7 cm in 3 s segment and partitions it into three equal parts, as shown in Figure 3]. Do you see how this is now

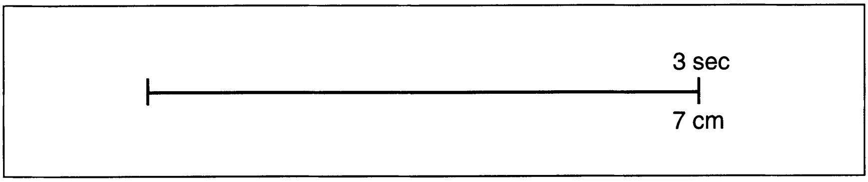

Figure 1. Teacher's diagram of a duck's journey of 7 cm in 3 s

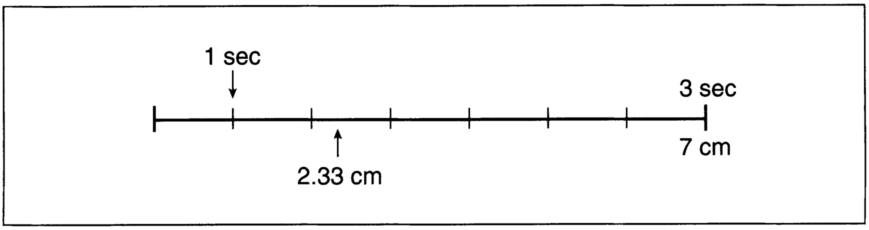

Figure 2. Carissa's attempt to show division by 3 in the diagram

split into three parts? This is one part [circles the first section from the left in the diagram], and one part [circles the second section], and one part [circles the third and final section]. One thing that dividing by 3 does is split this number line or drawing into three equal parts. How far had you gone and in how many seconds in each part?

Carissa: The first part would be 2.33 cm and 1 s.

Teacher: Exactly. See this [points to the first section in the diagram] is one third of this whole 3 s, and one third of 3 s is 1 s, and one third of 7 cm is 2.33 cm. Does that help you see where the dividing by 3 is?

Carissa: Yeah.

Teacher: Your strategy was really good. I just wanted to get it connected with the picture so you could see why you were dividing.

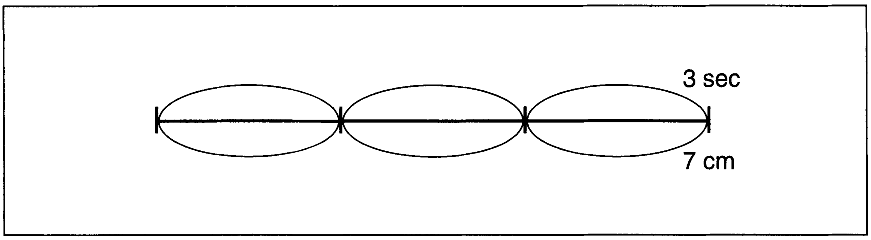

Figure 3. Teacher's representation of the connection between dividing by 3 and partitioning the segment into three equal parts

Because our goal is to reformulate telling by examining its *function* in the instructional environment, we examine the teacher's *intention* and *action* for the initiating act as well as the student's *interpretation* of the action. The last line of the excerpt provides some evidence that the teacher's intention was to help Carissa connect division with partitioning as opposed to making a decision to divide based on number size alone, as Carissa had done in the interview. The content of the "telling" act was conceptual rather than procedural in nature. The teacher did not provide Carissa with a step-by-step method for determining which number to divide by; instead, she presented an idea regarding the connection between partitioning and dividing. For the particular initiating action, the teacher employed both targeted questions and declarative statements. Despite the fact that the teacher posed some questions, we maintain that she was initiating, not eliciting.

The teacher believed that Carissa would be able to engage with the content of the initiating act for the following reasons. Carissa had been able to generate an appropriate word problem for $6 \div 3$ by divvying six children among three parties, which suggests an understanding that partitioning into three equal groups is connected with division by 3. When asked what division means in real life, Carissa had responded, "If you go to a closer store that is halfway to a farther store, you're dividing by 2," which indicates some understanding of the relationship between partitioning and division in a distance model. This suggests that Carissa may have reflected upon previous partitioning activities and formed a relationship between partitioning and dividing that she could unpack and use in the speed situation. If Carissa had those structures in place, and if she had sufficiently reflected on those ideas, then telling her about the meaning of dividing in the speed situation might allow her to make the connection back to her existing structures. To explore the function of the initiating action, we turn to an examination of Carissa's interpretation of the action.

Eliciting. The teacher did not assume that Carissa had constructed the same meaning as the teacher's. Consequently, she elicited Carissa's interpretation of the initiated idea by posing a carefully designed task (Henningsen & Stein, 1997; NCTM, 1991). The teacher drew a segment to represent a frog walking 16 cm in 4 s (Figure 4), told Carissa that she cut the segment into eight equal-sized sections, and then asked how far and for how long the frog walked in the circled section of the drawing. The teacher partitioned the segment into some number of sections other than four, because she wanted to know whether Carissa would calculate $16 \div 4$, indicating a strategy of operating on the two numbers provided in the problem statement, or whether Carissa would calculate $16 \div 8$, indicating an ability to perceive the line as partitioned into eight equal parts and an ability to connect partitioning by 8 with dividing by 8.

Carissa calculated $16 \div 4 = 4$ and $4 \div 4 = 1$ and then wrote 4 cm in 1 s near the first tick mark in the drawing. She iterated the "4 cm in 1 s" unit three times (see Figure 5). When she found that she had already reached 16 cm in 4 s by the halfway mark in the drawing, she knew something was wrong and gave up, complaining that she did not understand.

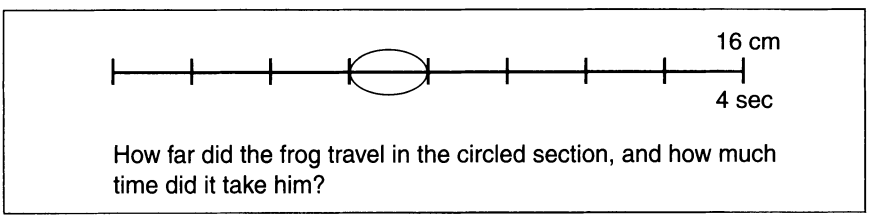

Figure 4. Task used to elicit Carissa's interpretation of the content of the teacher's initiating action

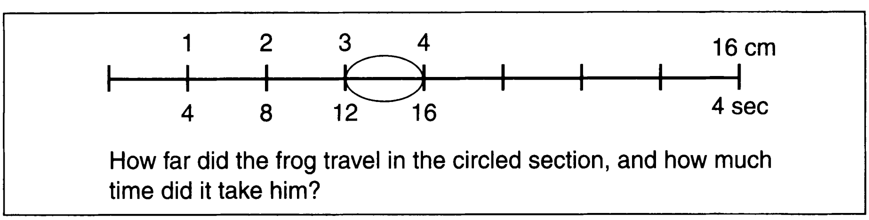

Figure 5. Carissa's response to the task posed in Figure 4

Determining the function of the initiating action. Discerning how the initiating action ultimately functioned in this instance is complicated. At the end of Session 9, the teacher repeated another E-I-E cycle with similar results. This suggests that, contrary to the teacher's hypothesis, Carissa did not possess the conceptual structures needed to actively construct with the information presented. However, this does not mean that Carissa did not engage with the substance of the initiating at some level. We provide a brief overview of what transpired over the next two sessions and then return to a discussion of the function of the initiating action.

In Session 10, the teacher elicited that Carissa had connected division by 2 with halving a line segment. The teacher built on this prior knowledge by asking Carissa to reflect on the effect of her activities of repeatedly halving the number line (and correspondingly dividing by 2) in order to develop a more general connection between partitioning and dividing. For example, the teacher drew a line segment representing 12 cm in 2 s, partitioned the segment into four equal parts and asked Carissa how far the character traveled and in how much time for one of the parts. Carissa calculated $12 \div 2$ and $2 \div 2$ to obtain 6 cm in 1 s, located 6 cm in 1 s at the midway point on the number line, and halved again to obtain the answer of 3 cm in 0.5 s. The teacher prompted Carissa to reflect on the effect of these two actions on the entire number line. Carissa appeared to connect partitioning the segment into 4 equal parts with dividing 12 cm by 4 and 2 s by 4.

During Sessions 10 and 11, the teacher relied primarily on eliciting actions. However, at the outset of Session 10 she initiated the general idea that arithmetic operations are connected with actions represented in drawings, without specifying any particular connection:

> Now what I am trying to help you do is connect these um. . . . These are called arithmetic operations—adding, subtracting, multiplying and dividing. These are things you calculate. And I want to help you connect those with your picture, because I think this will really help you.

This initiating act set the stage for the teacher to ask the following types of questions after each activity: "How many parts did you cut your line into?" "How is that connected to the calculation you performed over here?" We hypothesize that by providing the information that a connection between partitioning and dividing exists, the teacher provided a conceptual space that Carissa could explore in a goal-oriented manner.

Carissa's work in Session 10 showed some engagement with the teacher's idea, and the acts of initiating in Sessions 9 and 10 did not appear to have shut down Carissa's own thinking. For example, Carissa responded to the task shown in Figure 6 by calculating $14 \div 2$ and $5 \div 2$, splitting the drawing into two equal parts, and labeling the midpoint as 7 cm in 2.5 s. She then tried to partition the first segment into 2.5 sections but abandoned that effort when her attempt resulted in three sections. She started over and said she should cut the segment into five parts. Instead she halved the line twice, thus showing the dominance of halving. When Carissa

Suppose the frog walks 14 cm in 5 sec. I want to know how far he walks in 1 s, but to figure this out, I don't want you to calculate anything yet. Just think about your picture. How could you split up this picture to help you find how far the frog goes in 1 s?

Figure 6. Task used to elicit Carissa's connection between partitioning and dividing

realized that she had only cut the segment into four parts, she created a new drawing, cut the segment into five parts, calculated 14 ÷ 5 to obtain 2.8, checked by iterating, and reported that the frog traveled 2.8 cm in 1 s. Thus, Carissa's work shows an emerging connection between partitioning and dividing as well as a reliance on her own halving-dominated strategies.

By the end of Session 11, Carissa no longer relied on halving and appeared to have formed a more general connection between partitioning and dividing. At the beginning of Session 12 (after a 3-week break), the teacher assessed Carissa's understanding by returning to the original interview question set in a speed context: Suppose the clown walks 16 cm in 24 s, how fast is he walking? Carissa demonstrated multiple solutions. She determined (a) that the clown walked 0.66 cm in 1 s, by finding 24 ÷ 24 and 16 ÷ 24; (b) that the clown took 1.5 s to go 1 cm, by finding 24 ÷ 16 and 16 ÷ 16; and (c) that 16 cm in 24 s also indicated the clown's speed. When asked to draw a picture to represent any of her responses, Carissa drew the diagram shown in Figure 7 and gave the following explanation, indicating that she had adopted the language of "cutting" and had formed a connection between partitioning and dividing.

Teacher: What did you do?
Carissa: I drew 16 lines.

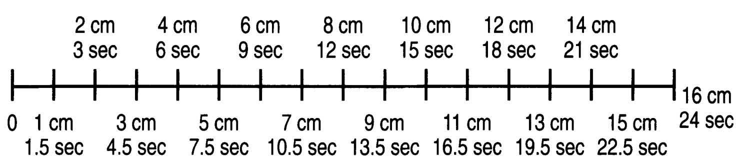

Figure 7. Authors' reconstruction of Carissa's diagram of a clown's journey of 16 cm in 24 s divided by 16

Teacher:	16 lines. Okay, what were you, um, what does the 16 lines mean?
Carissa:	That I was dividing it by 16.
Teacher:	Okay, what is it that you were dividing?
Carissa:	Um, I was dividing by 16 to get the centimeter.
Teacher:	What does dividing by 16 mean?
Carissa:	That, um, I'm going to cut it up into 16 parts.
Teacher:	Into 16 parts?
Carissa:	Yeah.
Teacher:	Okay, so what else can you tell me about your drawing? What do the little parts represent?
Carissa:	Um, for 1 cm it takes 1.5 s.

One can legitimately argue that the teacher could have supported Carissa's conceptual development without initiating. However, had Carissa been ready to hear, as the teacher hypothesized, then it would have been cumbersome and unnecessary to insist that she remain at a concrete activity level if she was ready to operate at a higher level of reflective abstraction. After conducting a retrospective analysis, we hypothesize that Carissa was unable to work with the substance of the first initiating act because the mental action that she re-presented to herself as she recalled prior division situations was that of divvying, which is different from the action of cutting. It is possible to conceive of centimeters as a discrete collection, and then divvy the centimeters among seconds. This means that one can "hand out" centimeters, one at a time, to a given number of seconds, in much the same way that one hands out cookies one at a time, cycling through the number of children until the supply of cookies is exhausted. However, it is more natural to rely on cutting actions. Thus, we hypothesize that Carissa needed to reflect on the effect of various cutting actions before she could imagine a line segment partitioned into a given number of parts and before she could reflectively abstract the connection between partitioning and dividing in the speed setting.

Despite the fact that Carissa did not engage with the original initiating action the way the teacher had hoped, we maintain that initiating was a reasonable action given the teacher's model of the student's understanding. Furthermore, it is plausible that initiating played a positive role in Carissa's conceptual development by creating a conceptual space in which Carissa could work to develop the particular connection. The uniqueness of Carissa's strategies in Session 10 indicates that initiating did not appear to harm Carissa's sense making. There are real risks associated with telling, but these risks arise in large part because telling has been conceived as an isolated expression of the teacher's knowledge. Episode 1 illustrates that by using eliciting in conjunction with initiating, many of these risks were avoided.

Episode 2: Initiating by Summarizing Student Work So That New Information Is Inserted

We present this episode as an example of initiating by inserting information (in this case, information about mathematical conventions) while summarizing student work (Action 2). The episode also demonstrates a pattern in which initiating occurs after a lengthy series of eliciting acts (an E- . . . E-I pattern). The episode is drawn from the last day of the summer teaching experiment during a 1.5-hour lesson designed to help students develop an understanding of slope as a ratio-as-measure of the steepness of a wheelchair ramp. Researchers have demonstrated that constructing slope as a measure of the steepness of a physical object such as a ski ramp or wheelchair ramp is difficult for secondary school students and preservice elementary teachers alike (Lobato & Thanheiser, 1999; Simon & Blume, 1994; Stump, 2001). Interviews with the participants before the teaching experiment indicated that they struggled with this construction as well.

Eliciting. The teacher began the lesson with the intention of gaining insight into why slope as a measure of the steepness of a physical object is difficult for students. In order to ascertain students' general comprehension of the situation as opposed to their calculational skills, she elicited by posing a task that contained no numbers. Specifically, she asked the students to draw two ramps, one steeper than the other, in order to elicit students' meaning for "steepness." This was an easy problem for students (see Jessica's drawing in Figure 8). When asked to describe steepness, students used words like "slantiness," "tilt," and "angle."

The importance of function: Unintentional initiating. The teacher intended to continue eliciting in order to locate the problem areas for students. The first sign of difficulty arose when the teacher pointed to one of the ramps that Jessica had drawn and asked if the ramp had the same steepness throughout or whether it was steeper in places. Six of the nine students reported that the ramp became steeper as they imagined climbing up the ramp. The following interchange illustrates one student's reasoning:

Teacher: Where is it steeper?

Terry: Like right there [to the far right of the ramp].

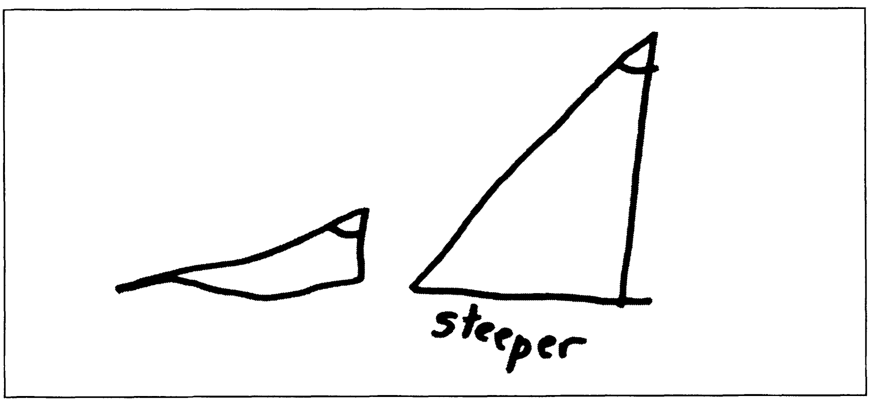

Figure 8. Jessica's drawing of two hills, in which one hill is steeper than the other

Teacher:	Okay, and how do you know it's steeper right there?
Terry:	'Cause it's like higher up on the angle.
Teacher:	Okay. So being higher up on the angle makes it steeper?
Terry:	Uh, huh.
Teacher:	So is the steepness constantly changing?
Terry:	Yeah.

The teacher's last utterance was intended to elicit by summarizing, but it goes beyond the student's own conceptions. In fact, this question may have functioned to initiate a new way to think about the ramp. At the time, the teacher did not realize that multiple plausible interpretations of Terry's utterance existed; for example, Terry could have meant that the ramp had the same steepness throughout but was steeper only at the rightmost point. It is easy for teachers to embed more sophisticated concepts (e.g., "constantly changing") in prompts intended only to elicit or clarify students' understanding. This unintentional initiating may be difficult to avoid. The value of sensitizing teachers to the initiating/eliciting distinction is that it focuses the teacher's attention on the conceptual sophistication of each statement rather than on its linguistic form (declaring versus asking). Because the teacher's priority is the elicitation of the students' conceptual understanding, he or she could be more likely to recognize inadvertent jumps in conceptual complexity as instances of unintentional initiating. This recognition may then trigger further eliciting to establish the students' interpretation of the initiating action.

Eliciting. Because the teacher thought the students were telling her the ramp kept getting steeper and steeper as one walked up, she posed the following task to perturb this conception: Draw two nonidentical ramps with the same steepness. This prompt can be described legitimately as eliciting since it employs a conceptual referent no more sophisticated than those already employed by the students. Furthermore, the prompt functioned to draw out more information about students' comprehension of the situation.

Students' disagreement over the solution to this problem revealed a great deal of information about which quantities they saw as affecting steepness. Jim correctly drew the picture shown in Figure 9, but Nathan disagreed, arguing that the ramp on the right was steeper because it was higher at one end. Josh also disagreed with

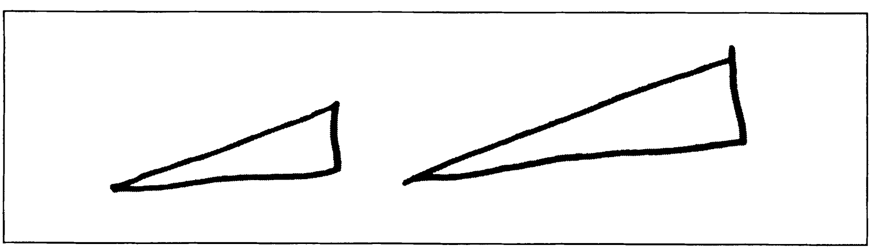

Figure 9. Jim's drawing of two nonidentical but equally steep ramps

Jim but for a different reason, offering that the part of the ramp that one walks up was longer for the ramp on the right. Josh may have understood steepness in terms of the length of the hypotenuse (if one thinks of the ramp as a triangle); thus, a longer hypotenuse implies a steeper ramp. Alternatively, he may have confused steepness with some other attribute, such as "work required to climb"; thus, in walking up a longer ramp, one is actually working harder.

When the teacher returned to Jessica's drawing (Figure 8) and asked the students what made the ramp on the right steeper, she elicited an informative response from Terry. He volunteered that the ramp on the right was steeper because it was higher. The teacher thought that it might be possible that the students assumed that taller ramps were always steeper. In order to find out, she asked the students to draw a ramp that was both shorter and steeper than either of Jessica's ramps.

A student's request for initiation. Before any of the students could respond to the task, Nathan requested that the teacher tell them what steepness means:

Nathan:	What's steeper?
Teacher:	That's what we have to decide together. There are a lot of quantities here and a lot of things that we are noticing.
Nathan:	But steepness, what does it mean?
Teacher:	That's what we have to decide together. Instead of me just telling you, I think you guys have ideas.

The teacher deliberately refused to define steepness at this point in the conversation for several reasons. She believed that the students had formed meaning for steepness, but was concerned that they may have entangled several other attributes with steepness. Recall that with the first task, students described steepness as "slantiness" and as "tilt." However, they also appeared to have linked steepness with the attribute of "work required to climb" and height. Thus, the teacher wanted the students to isolate these multiple attributes before she conveyed the conventional knowledge that steepness refers to the slantiness of the object. The teacher's decision not to initiate at this time is an important one since the student explicitly recognized and communicated a need to know (Clarke, 1994). In this case, the teacher was aware of other considerations requiring further elicitation before any initiating could usefully occur. The next task appeared to accomplish this goal of isolating height from steepness for many of the students.

Eliciting. Terry tried to draw a ramp that was shorter but steeper than the given ramps but was unsuccessful (see the ramp on the far left in Figure 10), thus suggesting that it may have been difficult for the students to isolate steepness from height. When the teacher asked the other students if Terry's ramp was steeper, Nathan blurted out, "Oh, I see what you're saying, like this is like going up, up, slowly up [gestures with his hand to illustrate an incline that is not very steep] and you can have a shorter one that goes shh shh [gestures with his hand to illustrate a steep incline with the first 'shh,' and then extends the incline with the second 'shh'] like that." Brad then successfully drew a ramp that was shorter yet steeper than one of the given ramps, and in the process seemed to isolate steepness from height. When

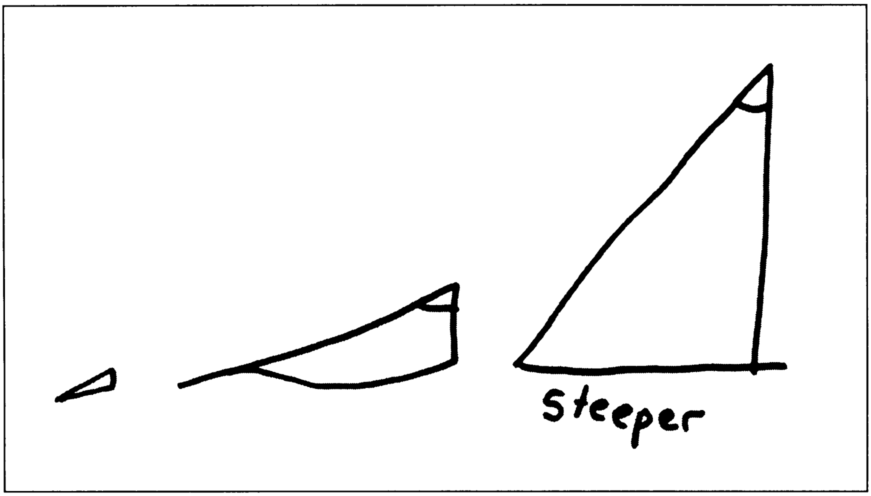

Figure 10. Terry's attempt to create a ramp that was steeper but not as high as the ramp on the far right

the teacher asked the students if they agreed with Brad, she noticed that more students seemed to be focusing productively on angles and "slantiness" in their responses. All of the students agreed that Brad's ramp was steeper.

Initiating. At this point in the lesson, the teacher decided to define steepness in terms of slantiness. Central to the decision to initiate is the establishment of conceptual readiness. The teacher felt that enough students had reflectively abstracted the attribute of slantiness as a distinct conceptual object from the attributes of height and work required to climb. As a result, the teacher believed that the students would be better able to make sense of the conventional characterization of steepness as slantiness. Had the teacher initiated earlier at Nathan's request, she may have afforded the continuation of the conflation of steepness with other attributes.

The teacher's initiating action took the form of asking students to summarize their work in such a way that she could introduce the link between steepness and slantiness. She began by asking the students to describe the differences they saw between the two ramps shown in Figure 9. The students reported that the ramp on the right was higher, longer, and harder to walk up. Then the teacher initiated the conventional knowledge that steepness refers to the slantiness of an object. She did this by inserting the idea as a culminating comment after the students had summarized the differences between the given ramps:

Teacher: The reason this is important is that there are important differences here between the ramps, and sometimes they get confused with steepness. But with steepness we're actually going to focus on something different. Um, do you think these have the same steepness [points to the two ramps shown in Figure 9]?

Students: [Silence]

Teacher:	Do they have the same slantiness?
Students:	Yes.
Teacher:	If I call this flat [gestures with forearm to show a horizontal "ramp"] and then a little steeper [gestures with forearm to show an incline that is nearly horizontal], steeper [gestures with forearm to show a steeper incline], steeper [gestures with forearm to show an incline that is nearly vertical]. This is getting steeper [sweeps arm from horizontal to nearly vertical]. Do these two ramps [pointing to Figure 9] have the same steepness?
Brad:	Yes.
Teacher:	Ok, right now we'll proceed but one thing I want to say is that when we're talking about steepness we're talking about this slantiness. We're not talking about whether it's harder to walk up it. They are definitely different ramps. You have to walk further on this one [points to the ramp on the right in Figure 9]. This one is higher. This one is longer. But there is something the same about them. And that's what . . .
Nathan:	. . . the slantiness. They're almost the same angle [gestures with his hand, tilting it up].
Teacher:	It has something to do with the angle, okay.

The remainder of the lesson contains several pieces of evidence that the students appeared to engage cognitively with the initiated ideas. For example, the teacher asked if someone could draw two nonidentical ramps with the same steepness. Jim said that the ramps shown in Figure 9 would work, and several students agreed. The teacher reminded them that at one point in the discussion, Nathan and Josh had argued that these ramps did not have the same steepness. Brad suggested that they could make the two ramps have exactly the same steepness if they had "more supplies," but that they were pretty close. The students seemed to now see the ramps in Figure 9 as having the same degree of steepness. Additionally, the students appropriately used the term steepness in the next 15-minute episode of the lesson in which they used a dynamic sketch in The Geometer's Sketchpad (Jackiw, 1995) to explore how different attributes affected steepness. Finally, none of the problems that had cropped up earlier for the tasks associated with Figures 9 and 10 appeared later in the lesson when the teacher asked the students to use The Geometer's Sketchpad to create a family of ramps with the same steepness. Although we cannot claim that the initiating action eliminated all confusion, for the remainder of the lesson the students appeared to focus on the attribute of steepness without conflating it with other attributes. Results from another study (Lobato & Siebert, 2002) further support this claim.

Episode 3: Providing Information So That Students Can Test Their Ideas

We present this final episode as an example of initiating by providing information to allow students to test their ideas or generate a counterexample (Action 3). The episode also illustrates a pattern in which initiating occurs near the beginning of the episode, E-(I&E). This episode is drawn from the same context as for Episode 2 but occurs later in the lesson. One of the teacher's goals was to help the students understand that the ratio of a ramp's height to its length could be a legitimate measure of its steepness. The episode involved the following problem situation:

> Suppose you work for a company that builds wheelchair ramps. You are in charge of making a catalogue for the different ramps that you make. You want to be able to communicate to people how steep the ramp is so they know what kind of ramp they will be buying. How would you measure the steepness of a ramp?

Eliciting. When the teacher asked for ideas about how to measure the steepness of a ramp, Denise suggested they measure the hypotenuse, which could be seen as a reasonable attempt to directly measure the part of the apparatus that is steep. Brad measured the length of the hypotenuse of one of the ramps that was on the board from the earlier discussion of steepness (see Figure 11), and reported that it was 6 in.

Initiating and eliciting. The teacher did not think the students would be able to devise a way to test Denise's conjecture, because they lacked the understanding about measures necessary to create such a test. In the following excerpt, the teacher provided some information about the nature of a measure, namely, that a measure of the steepness of a particular ramp should allow someone else to determine its steepness:

Teacher: Now we have to talk about what a measure is. Could this [referring to the length of the hypotenuse] be a measure of steepness? By measure of steepness, what I mean is if a construction worker or someone who builds wheelchair ramps says, "I have a wheelchair ramp with a hypotenuse of 6 in." Does that tell you how steep it is?

Brad: No.

Teacher: Brad says no, Jessica says no. Jim says yes.

Denise: Can you say that again?

Teacher: What we're trying to do is come up with a number that we can tell people and when they hear it, they will know how steep the ramp is.

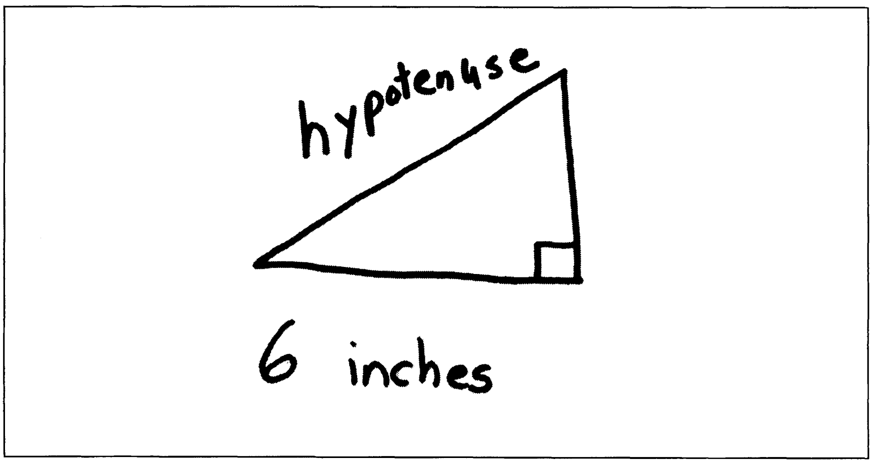

Figure 11. Brad's attempt to create a measure of steepness of a given ramp

Denise:	Oh, okay.
Teacher:	Would 6 in. be it?
Brad:	No.

Denise's request for the teacher to repeat this information suggests that she and perhaps others needed more information about what a measure was supposed to convey. The teacher decided to continue initiating by taking on part of the task for the students (namely, providing information about what constitutes an appropriate test of a measure) while eliciting their ideas about which quantities to test as a measure of the steepness of ramps. Specifically, the teacher attempted to convey the information that if Denise's measure of 6 in. was a good measure of steepness, then every ramp that the students could draw with a hypotenuse of 6 in. should have the same steepness:

> To test whether this 6 in. could be a measure, I'm going to make Josh and Jessica test this because they both think that it isn't [a measure]. I want both of you to go up to the board and see if you can make a ramp, and the rest of you can be making one on your sheets of paper, that has a hypotenuse of 6 in. that is not as steep or is steeper than the one on the board.

When the students carried out this activity, they discovered that some of their ramps were steeper than others. Thus, they concluded that the length of the ramp alone was not a good measure of steepness. The students repeated this process twice more, each time suggesting, testing, and eventually rejecting different extensive measures,[4] namely, the height of the ramp and the length of the base of the ramp. They also conjectured, tested, and accepted the angle of inclination formed between the hypotenuse and the base of the ramp as a good measure of its steepness. The majority of the students eventually constructed an intensive measure, namely, the ratio of the height of the ramp to the length of the base (see details in Lobato & Thanheiser, 2000, 2002).

The influence of initiating. The students' ability to test various quantities as measures provides some evidence of engagement with the initiated ideas about the nature of measures. Furthermore, some students appeared to use some of the ideas introduced by the teacher. For example, after conjecturing whether the height could be a reasonable measure of the steepness of a ramp, Katie asked what they were supposed to do (presumably to test the measure). Denise explained, "We are supposed to draw ramps with a height of 5 3/4 in. to see if we all get the same steepness," thus indicating some engagement with the teacher's initiated ideas about the nature of measures.

What is perhaps more important is that we do not see evidence that the initiating action limited cognitive engagement on the part of the students other than being relieved of the burden of figuring out a way to test their measures. If the students

[4] Schwarz (1988) distinguished *extensive* quantities, such as distance or age, which can be counted or measured from *intensive* quantities, such as speed or gas efficiency, which are composed as a ratio.

were capable of finding a way to test their measures, then eliciting actions may have been a better choice. Despite the fact that part of the task was aided by the teacher, the conversation indicated that there was more than one possible solution path validated in the classroom. Students presented a variety of ideas, unanticipated by the teacher, that were uniquely theirs. Although it is possible that premature closure of mathematical exploration occurred, the teacher's decision clearly led to a situation in which the students engaged in a rich and lengthy investigation of important mathematical ideas.

It is possible that by providing students with a way to test their results, some authority was relegated to the teacher. This could have circumvented the development of student responsibility for judgments of mathematical correctness and coherence. However, it is also possible the students simply did not understand enough about the characteristics of measures to find their own ways to test the validity of their ideas. The teacher believed that asking students to make conjectures was a sufficiently challenging task without additionally requiring them to devise a way to check their conjectures. In fact, by carrying out the tests of each potential measure and deciding in the end whether or not it was a valid measure of steepness, the students did share in the responsibility for the judgments of mathematical correctness and coherence. Weighing such competing factors constitutes the essence of reflective pedagogical action.

DISCUSSION

Our intention in this article was to move away from a false choice between constructivist teaching approaches and telling methods toward a more sophisticated range of pedagogical actions. Rather than simply including judicious telling into the palette of teaching actions, we have argued instead for a rethinking of the conceptual roots of telling. Whereas others have rethought telling and expanded its definition (Chazan & Ball, 1999; Hiebert et al., 1997), we have addressed the action typically left out by researchers—direct introduction of new information and ideas. More important, our approach shifts the focus from the form of a teacher's action to its function in the classroom. By doing so we emphasize the need to attend to how a telling act is interpreted by students. Rather than making assumptions about which teacher actions will appropriately problematize mathematics for students, the shift toward function addresses pedagogical acts from a student-centered view.

The other important distinction of our approach is its incorporation of telling actions into a larger pedagogical structure. By viewing initiating and eliciting as mutually informing one another, we help guard against some of the legitimate concerns raised with telling. Our consideration of initiating as part of a larger structure focuses attention on the interaction between teacher and students in a manner that consistently attaches priority to the development of the students' mathematics, rather than to the communication of the teacher's mathematics. This allows us to focus on telling as an act of communicating conceptual rather than procedural content. Other research on expanding telling actions has focused on "how to tell" in a manner that does not *impede* students' conceptual growth. We instead consider how one can exploit telling actions to *promote* conceptual growth by linking initiating to the process of reflective abstraction.

Although traditional telling is grounded in a broadcast/reception model of communication—a model that has been properly rejected by both researchers and teachers in mathematics education—we instead argued from our data that communication constitutes a state of dialectical tension between the teacher and the learner. This model of linguistic behavior is closer to emerging models in which words are seen not as "things we move from one place to another," but as "tokens for linguistic coordination of actions" (Maturana & Varela, 1992). Viewed from this perspective, removing one of these actions from its context and labeling it as an example of telling violates the linguistic give-and-take between the student and teacher over meaningful mathematical content. To then place this context-less utterance at odds with a constructivist way of coming to know is to create a category error in which a framework for *understanding* is pitted against a class of *speech acts* (Searle, 1969).[5] We resolve this problem by examining the processes of telling empirically, showing that the definition of what constitutes telling is context sensitive. This is why we attend to the teacher's intentions and actions as well as to the students' interpretation of the initiated idea for each act of initiation.

We acknowledge that there are multiple sophisticated teacher actions beneath the umbrella term "telling," and that these initiating actions differ significantly across the three instructional episodes analyzed in this paper. The action of declaring a connection between partitioning and the calculation of division in Episode 1 may be perceived as most similar to traditional telling. The main difference is that the conveyed information was conceptual rather than procedural in nature. However, in Episodes 2 and 3, the teacher did not tell students the main idea that she wanted them to understand; instead, she provided ancillary information that afforded their mathematical investigation. Furthermore, the initiating actions in Episodes 2 and 3 involved different forms of utterances than declarative statements and different relationships between initiating and eliciting. In short, the ways in which we reformulated telling instantiated themselves differently across the three empirical episodes.

The data indicate that different local intentions relate to different interactional patterns of initiating and eliciting. In Episode 1, the teacher's intention was for the student to develop a connection between partitioning and calculating. Thus, a cyclic pattern of initiating and eliciting was appropriate. The teacher preceded initiating with eliciting so that she could gather information about the student's thinking before making a judgment to introduce new information. Once the teacher engaged in initiating, she then stepped back and elicited to see what the student did with that information. In Episode 2, the teacher's intention was to introduce a mathematical

[5] The authors thank Anthony E. Kelly for bringing to their attention this connection to Searle's work on speech acts.

convention linking steepness with slantiness. However, she felt that the students would be better prepared to make sense of this information if they first isolated the attribute of slantiness from other attributes in the situation. Thus, a lengthy period of eliciting preceded the initiating. Finally, in Episode 3, initiating appeared near the beginning of the episode because the teacher's intention was to relocate part of the task onto herself in order to enable students to engage in exploration. These links between different intentions and related patterns of initiating and eliciting can be applied to a range of instructional settings.

We recognize that initiating and eliciting are only two ends of a spectrum rather than a full set of pedagogical actions. Further research is needed to articulate other actions that could contribute to the system of actions related to initiating and eliciting. We also acknowledge that there are other legitimate ways to conceive of the pedagogical actions developed in this article. However, our goal in defining initiating as serving the function of introducing new information was to validate the part of telling that has historically been most difficult to reconcile with constructivism. We hope that by articulating initiating and eliciting, we can move beyond the unnecessary tension between constructivism and telling to a recognition of the much richer and more subtle range of actions available to teachers seeking to promote conceptual development.

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