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PK 12

Why Multiply? Area Measurement and Multiplicative Reasoning

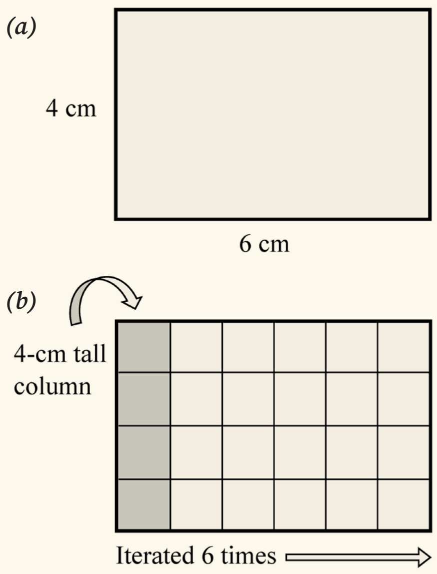
Asked to quantify the changes in area of growing rectangles, these students reasoned about multiplicative relationships in interesting new ways.

Brandon K. Singleton and Amy B. Ellis

Think about a rectangle that is 6 cm long and 4 cm tall (see figure 1a). Why can the rectangle's area be found by multiplying its length by its width? Despite our best efforts, helping children understand *why we multiply* to measure area is an enduring problem. Many students can recite the $A = L \times W$ formula, but they often do not understand it. Making sense of arrays of unit squares is one basis for understanding the transformation of length measurements into area measurements. For instance, for a 4-cm by 6-cm rectangle, students can think about the area as measured in square units, and the total number of square units, 24, can be found by imagining a column of 4 square centimeters iterated 6 times (see figure 1b).

Some students do develop this meaning. For instance, Anya, a sixth grader, calculated the area of a 1.5-cm by 4-cm rectangle and explained that multiplying made sense because there would be 1.5 rows and 4 columns: "That creates sort of, like, squares or

Fig. 1



For a 4-cm by 6-cm rectangle partitioned into columns and rows, students can think about measuring the area in square units (a); and finding the total by imagining a column of 4 square centimeters iterated 6 times (b).

rectangles within the rectangle, and there's 4 rows of 1.5. So that would be like 4 times 1.5, so that's how we figure out how many, or how the area would be."

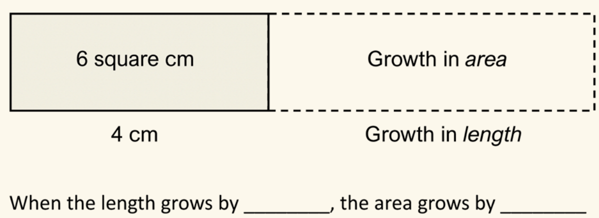
Not all students, however, are able to make sense of the row-column array structure in the way Anya did. For this reason, Battista and colleagues suggest that typical instructional treatments of area and multiplication should be rethought: "If students do not see a row-by-column structure in these arrays, how can using multiplication to enumerate the objects in these arrays, much less using area formulas, make sense to them?" (Battista et al. 1998, p. 531). Students may not naturally group the unit squares into rows and columns, and if they do, they may not associate the number of rows and columns with corresponding side measures. Moreover, imagining partial rows and columns is difficult for rectangles with noninteger dimensions. Many students we observed did not make the argument Anya provided for the 1.5-cm by 4-cm rectangle.

We designed an activity called the Growing Rectangle problem that connects multiplication with area measurement in a new way. Inspired in part by a task given by Johnson (2013), the Growing Rectangle problem has three key features:

1. Areas are calculated by imagining a *transformation* rather than by measuring a fixed object.
2. The setup provides a *length-area pair* instead of a length-height pair.
3. The prompt allows for *multiple initial* responses as students explore how changes in length and area occur together (see figure 2).

We used the Growing Rectangle problem with 13 middle-school students in grades 6–8. Although nine students eventually used the area formula, most did not think of the formula immediately. Instead, the students had to grapple with how length and area change together. We share the thinking of several students below.

Fig. 2



The Growing Rectangle problem

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OLIVIA: UNIT RATE

Olivia, a seventh grader, filled in the blanks with 2 and 3 and explained, "I know that 2 is half of 4, so then I did half of 6." Olivia's picture in figure 3 shows a growth of 2 cm in length and 3 cm^2 in area. She also developed other pairs, such as 3 cm: 4 1/2 cm^2, 1 cm: 1 1/2 cm^2, and 16 cm: 24 cm^2. In explaining her work for the 3 cm: 4 1/2 cm^2 pair, Olivia said, "I reduced it to be, like, the unit rate of how much it would go up by. So, if I know that 1 [centimeter in length] would be 1 1/2 [centimeters squared in area], then I times that by 3, to get 4 1/2." Olivia recognized that the rectangle gained area at the rate of 1 1/2 cm^2 for every additional 1 cm of length, and she used that rate to find other pairs.

Olivia could also express this relationship as a general formula in terms of expressing the growth in area for an *x*-cm growth in length. She wrote "1.5*x*" and explained, "You can replace *x* with anything." Olivia did not think about 1.5 as the rectangle's height at this point; instead, it was a unit rate expressing how much area the rectangle gains per 1 cm of length as it grows (see figure 4).

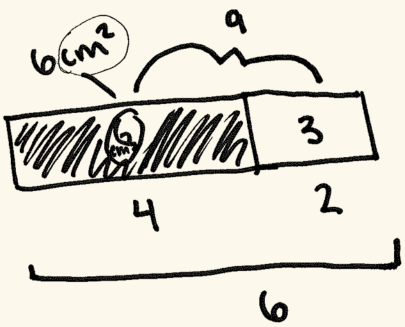
Fig. 4

$$A = 1.5 \cdot x$$

Area	Rate	Length
cm^2	cm^2 per 1 cm length	cm

The meaning for Olivia's area formula

Fig. 3



Olivia's picture of the rectangle growing by 2 cm in length and 3 cm^2 in area.

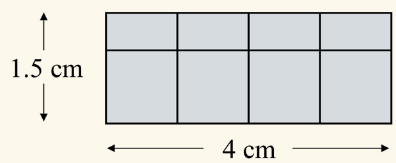
CONNECTING MULTIPLICATION TO AREA UNITS

In geometric measurement, it is important to understand what the unit of measure is. The Growing Rectangle problem gives measurements, but it does not indicate what the units that were used for measuring look like. Students must bring their own meanings to the problem to infer what a square centimeter is and where the 6 cm^2 are in the figure. Teachers can help students make their ideas explicit and think more purposefully about units of measure. We asked students to show where the 6 cm^2 were in the rectangle. This can be challenging when using unit squares because the height is not a whole number. It requires conceiving of the 1.5-cm height as 1 square centimeter and 1/2 of a square centimeter, iterated 4 times to create 6 square centimeters (see figure 5a).

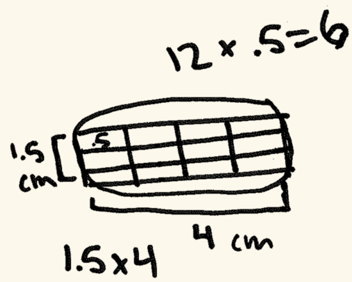
In Olivia's case, she partitioned the rectangle into 4 columns and 3 rows to make 12 rectangular cells. She understood that each cell was 0.5 cm tall to make an area of 0.5 cm × 1 cm = 0.5 cm2 (see figure 5b), explaining, "The height of each of these little boxes would be 0.5. And so then, you'd just go 0.5, 1, 1.5 (the entire column), and then times 4." However, Olivia was unable to relate the cells in her drawing to square centimeters. When

Fig. 5

(a)



(b)



The 1.5 cm by 4 cm rectangle is partitioned into (a) square units; and (b) rectangular cells.

the interviewer, said, "I see 12 rectangles," and asked her, "Can you show me where the 6 would be then—the 6 square centimeters?" Olivia responded, "Well, this is kind of the 6. So, the whole thing" [she circled the entire rectangle].

Although Olivia appeared unfamiliar with the convention that 1 cm^2 is a 1-cm by 1-cm square, she showed flexibility in partitioning the rectangle and in identifying the correct size or amount that corresponded to her partition. In many tasks, students are provided with square units, such as tiles, and are asked to measure areas. They do not have to think about where the unit tiles come from. In contrast, the Growing Rectangle problem provides students with an area measure and challenges them to think about what 1 cm^2 represents. By using partitioning strategies, students can come to appreciate that a multiplicative relationship exists between the amount contained in one unit and the amount contained in the whole measure. Olivia was able to relate the amount in one of her cells to the whole rectangle by using multiplication, writing the 6-cm^2 area both as 1.5×4 and as $12 \times .5$ (see figure 5b).

SPIKE: PROPORTIONALITY VERSUS SIMILARITY

Like Olivia, Spike (a seventh grader) believed that the length and area were proportional. He found new pairs of length and area by setting up proportions and solving with cross products. However, Spike became uncertain about his answers because he realized that the growing rectangle did not maintain similarity as it grew. Spike confused similarity with proportionality, stating, "The length is expanding, but the width isn't; so that wouldn't make it proportional. That would make it a different shape." Because the rectangle did not maintain similarity, Spike came to believe that all of his answers were wrong. When he checked his answers using the area formula, he was astonished that they matched: "Oh that was right. Huh, that's weird . . . . I'm getting the same answers as I had before."

Spike is not alone in confusing proportionality and similarity. In another study, students solved a problem about a window painting of Father Christmas (see figure 6) that was to be scaled up by a factor of three (Van Dooren et al. 2003). When asked how much paint would be needed, almost all the students multiplied the original amount by three instead of nine. The students could not explain why they believed a proportional strategy worked, stating, "I don't know. I just solved it that way," and "It just works. I don't know why" (pp. 206–7). Although Spike assumed that a proportional strategy must imply similarity, the students in the study that Van Dooren and colleagues conducted (2003) assumed that similarity must imply a proportional strategy.

Both the Father Christmas problem and the Growing Rectangle problem can cause confusion in students who have come to think of similarity and proportionality as two sides of the same coin. Typical area tasks may contribute to this confusion if all the proportion problems from geometry rely on similar figures. The Growing Rectangle problem, in contrast, provides an opportunity to work with proportionality in a geometry setting that does not rely on similarity. Teachers could use the Growing Rectangle problem as a bridge to making sense of similarity by breaking the scaling transformation into two steps. Each step involves scaling the area in one dimension by a proportional factor k, with the final result being an increase by a factor of k^2. (View video 1 for further illustration of scaling in two dimensions.)

Fig. 6

Bakery's door **Supermarket window**

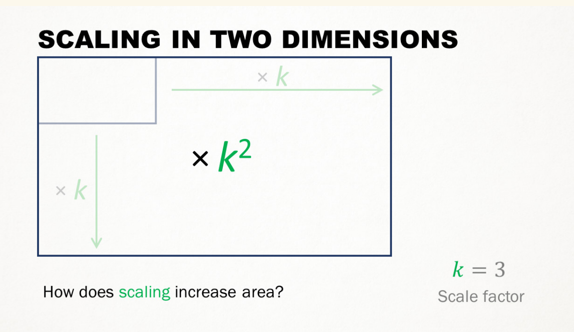
Bart is a publicity painter. In the last few days, he had to paint Christmas decorations on several store windows. Yesterday, he made a drawing of a 56 cm high Father Christmas on the door of a bakery. He needed 6 ml of paint. Now he is asked to make an enlarged version of the same drawing on a supermarket window. This copy should be 168 cm high. Approximately how much paint will Bart need to do this?

The Father Christmas problem. From "Improper Applications of Proportional Reasoning," by W. Van Dooren, D. De Bock, L. Vershaffel and D. Janssens, 2003, Mathematics Teaching in the Middle School *9, no. 4, p. 205. Reprinted with permission.*

CHALLENGES IN SUPPORTING MULTIPLICATIVE REASONING

Not all students to whom we gave the Growing Rectangle task related length and area multiplicatively. Four students approached the task using additive rather than multiplicative reasoning. For example, Willow, a sixth grader, wrote, “If the length and the area grow evenly (by the same amount), the area will always be 2 more than the length.” Willow declared the area to always be 2 cm greater than the length, regardless of what the length would be. Thus, she identified a constant *difference* between length and area rather than a constant *ratio*. If Willow had visualized a very long rectangle, such as a 20-cm rectangle or a 100-cm rectangle, she might have realized that the corresponding area would need to be larger than 22 cm^2 (or 102 cm^2). However, it is also possible that Willow was not imagining the rectangle’s area in terms of units of measure, so this type of visualization may not have prompted her to realize that the relationship between length and area could not be additive. Another approach could be to ask Willow to double the original 4-cm long rectangle. If she drew a second rectangle next to the first one (as outlined in the dashed version in figure 2), Willow could be asked to describe the area of the second (identical) rectangle with an additional length of 4 cm. This rectangle, being a copy of the first, would have 6 cm^2 for area, which might help Willow see that the total rectangle that was 8 cm long would need to have an area of 12 cm^2.

Video 1 Scaling Two Dimensions



Watch the full video online.

Another student, Angelo (an eighth grader), also initially relied on additive reasoning, stating that a rectangle with a length of 6 cm would have an area of 8 cm^2. Angelo’s answer, although incorrect, was based on trying to visually estimate the size of the dashed region in figure 2. In this case, posing doubling and halving scenarios helped Angelo shift to a multiplicative strategy. For example, given a growth in length of 4 cm, Angelo realized, “I think that it would grow by 12, because is 4 is doubling, and if that’s 4 centimeters [the original length] and that’s 4 centimeters [the growth in length], then that [total length] would equal 8 centimeters. And then another 6 centimeters [squared, in area] would equal 12 centimeters [squared, for the total area].” Then, when asked to imagine the rectangle growing by half its amount (another 2 cm), Angelo explained, “I think it will grow by 3 [cm^2], because 4 [cm] divided by 2 is 2 [cm], and then 6 [cm^2] divided by 2 is 3 [cm^2].” Students such as Angelo and Willow can benefit from strategies recommended for fostering initial ratio reasoning, which include attending to both quantities (length and area) together; making drawings of different-sized rectangles together with tables to keep track of length and area pairs; beginning with simpler cases such as doubling, quadrupling, and halving; and gradually moving to more difficult multipliers (Lobato and Ellis 2010).

The Ratio-And-Proportion Meaning

Because students often rely on the area formula without understanding why they should multiply, we designed the Growing Rectangle problem to encourage ratio reasoning. The Growing Rectangle problem affords a new way of explaining why we multiply when calculating area, and it helps students connect proportional reasoning to area measurement. Most problems about area in the curriculum involve static shapes, and students calculate areas by counting how many unit tiles cover the region until they understand row-column arrays and derive the formula $A = L \times W$. Students justify the formula’s multiplication operation with an equal-groups meaning or an array meaning. The Growing Rectangle problem leverages a different meaning of multiplication, the ratio-and-proportion meaning. This meaning does not lend itself as readily to the $A = L \times W$ formula, but it can lead to an algebraic equation based on rate thinking. Students use multiplication directly and spontaneously to calculate changes in area. They iterate and partition ratio pairs just as they would for other equal-ratio problem contexts. Multiplication can supply a ratio meaning to the units of measure so that a

6-cm^2 object is six times as large as the unit, 1 cm^2.

As students reason about the Growing Rectangle problem, we should use caution when interpreting their proportional strategies. As we saw with Spike, students may use proportions computationally before they can justify doing so conceptually. Willow and Angelo tried out multiplicative strategies only after attempting additive ones, and it was not obvious to them at first which strategy was correct. Determining why multiplicative reasoning is appropriate on the basis of the problem's geometric relationships is important for students. Asking them to justify their strategies can help.

The Growing Rectangle problem is a challenging, nonroutine task that can promote problem solving. Strategic implementation can help individualize the task to each student's strengths and needs. See the sidebar for some implementation tips. With the appropriate supports, students can develop multiplicative strategies that they can understand, justify, and eventually connect to the area formula.

Implementation tips for the Growing Rectangle problem

• **Posing:** You can leverage the value of the task by posing sequences of questions that respond to students' thinking and lead students forward. Use students' initial responses to guide next steps and manage the problem's difficulty. You can pose different values in one blank and request the missing value. Choose numbers purposefully and think about the relationships involved in moving from one number pair to the next. Skillful sequences will help your students confront and overcome challenges. You can also extend the task with new shapes, like a growing parallelogram.

• **Ratio:** This task can help students build their ratio reasoning. Some students may need to start with easier numbers that rely on doubling or halving to create ratios. Once they can do that, you can pose values that help students build up more complicated ratios. Over time, students should develop a unit ratio and express the relationship between length and area algebraically.

• **Visuals:** In addition to providing students with pictures, consider showing the situation dynamically with technology and allowing students to drag the rectangle to see its growth. Also, ask students to draw their own figures and show the quantities they calculate in the figures they have drawn.

• **Quantities:** Make sure that students do not lose track of the quantities in the situation, such as length, height, and area. Ask them to refer to the specific units they have calculated so that they can keep track of the relationships between the associated quantities. You may need to help students distinguish carefully between added growth (such as added length and added area) and total growth.

REFERENCES

Battista, Michael T., Douglas H. Clements, Judy Arnoff, Kathryn Battista, and Caroline Van Auken Borrow. 1998. "Students' Spatial Structuring of 2D Arrays of Squares." *Journal for Research in Mathematics Education* 29, no. 5 (November): 503–32.

Johnson, Heather Lynn. 2013. "Informing Practice: Predicting Amounts of Change in Quantities." *Mathematics Teaching in the Middle School* 19, no. 5 (December): 260–64.

Lobato, Joanne, Amy Ellis, and Rose Mary Zbiek. 2010. *Developing Essential Understanding of Ratios, Proportions, and Proportional Reasoning for Teaching Mathematics: Grades 6–8*. Reston, VA: National Council of Teachers of Mathematics.

Van Dooren, Wim, Dirk De Bock, Lieven Verschaffel, and Jirk Janssens. 2003. "Improper Applications of Proportional Reasoning." *Mathematics Teaching in the Middle School* 9, no. 4 (December): 204–9.