Chapter 8

How Generalizing Can Foster Proving and Vice Versa: A Case with Linear Functions

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Generalization and justification are critical components of mathematical reasoning, with researchers and reform initiatives advocating a central role for generalization and proof at all grade levels (National Governors Association Center for Best Practices and Council of Chief State School Officers [NGA Center and CCSSO] 2010); Knuth, Slaughter, Choppin, & Sutherland, 2002; National Council of Teachers of Mathematics [NCTM], 2000). These recommendations have led to a wealth of studies on how to foster generalizing and proving (e.g., Ellis, 2007b; Knuth, Choppin, & Bieda, 2009; Stylianides, 2007), as well as curricula geared towards supporting these activities (e.g., Lappan et al., 2006). Research examining students' abilities to generalize and justify, however, reveal difficulties in both recognizing and creating correct general statements and proofs (Chazan, 1993; English & Warren, 1995; Kieran, 1992; Knuth et al., 2002). Although these challenges are well documented, few studies have devoted significant attention to the interplay between generalizing and justifying, despite the fact that sophisticated mathematical reasoning depends on deep involvement in both activities. How students generalize influences the tools they bring to bear when justifying their generalizations, and working to prove a statement could similarly influence students' subsequent generalizing activities.

Because both generalizing and justifying can influence the development of the other, it is important to understand the connections between these two activities. This chapter reports on a study of seventh-grade students' generalizations and justifications as they learned about linear relationships. It identifies four mechanisms of change for promoting increased sophistication in students' generalizing and proving, and it shares data illustrating the connections between them.

What Is Generalization and Proof?

What does it mean for a student to generalize or produce a generalization? In the spirit of Kaput's (1999) view, I define generalization as engaging in at least one of three activities: (*a*) identifying commonality across cases, (*b*) extending one's reasoning beyond the range in which it originated, or (*c*) deriving broader results from particular cases. This definition relies on Lobato's (2003) actor-oriented perspective, which requires the observer to abandon any predetermined idea of what should count as a generalization and instead to seek to identify how students generate their own similarity relations. Rather than asking, "Did the student produce the generalization [that I was hoping to see]?", we ask, "What does the student see as general in this problem or similar across these problems [even if it is not what I was hoping for]?" This stance acknowledges the importance of mathematical correctness but also values the need to understand what students themselves see as general.

In prior research I developed a taxonomy of the different types of generalizations middle-school students created (Ellis, 2007c). The taxonomy distinguishes between students' activity as they

This chapter is adapted from Ellis, A. B. (2007a). Connections between generalizing and justifying: Students' reasoning with linear relationships. *Journal for Research in Mathematics Education*, *38*, 194–229.

generalize, called generalizing actions, and students' final statements of generalization, called reflection generalizations. Generalizing actions (see fig. 8.1) fall into three major categories. When *relating,* students form an association between two or more problems, situations, ideas, or mathematical objects. They relate by recalling a prior situation, inventing a new one, or focusing on similar properties or forms of mathematical objects. When *searching,* students engage in a repeated mathematical action, such as calculating a ratio or locating a pattern, in order to find an element of similarity. Students focus on relationships, procedures, patterns, or solutions when searching. Finally, *extending* involves the expansion of a pattern, relationship, or rule into a more general structure. Students who extend widen their reasoning beyond the problem, situation, or case in which it originated.

Type I: Relating		Examples
Relating situations: The formation of an association between two or more problems or situations	*Connecting back:* Connecting between a current and previously encountered situation	Realizing that "This gear problem is just like the swimming laps problem we did in class!"
	Creating new: Inventing a new situation viewed as similar to an existing one	"He's walking 5 cm every 2 s. It'd be like a heart that was beating at a steady pace, 5 beats in 2 s."
Relating objects: The formation of an association of similarity between two or more present objects	*Property:* Associating objects by focusing on a property similar to both	Noticing that two equations in different forms both show a multiplicative relationship between x and y
	Form: Associating objects by focusing on their similar form	Noticing that "Those equations both have one thing divided by another"
Type II: Searching		**Examples**
Same relationship: Performing a repeated action in order to detect a stable relationship between two or more objects		Dividing y by x for each ordered pair in a table to determine if the ratio is stable
Same procedure: Repeatedly performing a procedure in order to test whether it remains valid for all cases		Dividing y by x as above without understanding what relationship is revealed by division; dividing as an arithmetic procedure to determine whether the resulting answer is the same
Same pattern: Checking whether a detected pattern remains stable across all cases		Given a table of ordered pairs, noticing that the y-value increases by 7 for each successive pair
Same solution or result: Performing a repeated action in order to determine if the outcome of the action is identical every time		Given an equation such as $y = 2x$, substituting multiple integers for x and noticing that y is always even
Type III: Extending		**Examples**
Expanding the range of applicability: Applying a phenomenon to a larger range of cases than that from which it originated		Having found that the difference between successive y-values is constant for $y = mx$ equations, applying the same rule to $y = mx + b$ equations
Removing particulars: Removing some contextual details in order to develop a global case.		Given that an increase of 1 for x results in an increase of 6/5 for y, and an increase of 1 for y results in an increase of 5/6 for x, developing a general description of this inverse relationship for all linear functions
Operating: Mathematically operating upon an object in order to generate new cases		Knowing that y increases by 6 for every unit increase for x, halving the (1:6) ratio to create a new ordered pair
Continuing: Repeating an existing pattern in order to generate new cases		Knowing that y increases by 6 for every unit increase for x, continuing the (1:6) ratio to create new ordered pairs

Fig. 8.1. Generalizing actions

Reflection generalizations are the general statements (written or verbal) students make after engaging in generalizing actions. This taxonomy appears in figure 8.2.

The actor-oriented framework can also inform ways of thinking about proof; one can define proving or justifying as a process of removing or creating doubts about the truth of an observation (Harel & Sowder, 1998). From this perspective, a researcher can try to understand what students view as convincing arguments. Harel and Sowder developed a framework to identify students' proof schemes, categorizing individual schemes of doubts, truths, and convictions. Five proof schemes from their framework applied to the students in the study reported here (see fig. 8.3). The first two proof schemes, *authoritarian* and *symbolic,* fall under the external conviction family. Under these schemes, conviction is obtained by the word of an authority, or the symbolic form of an argument. Under the empirical family of proof schemes, conjectures are validated or invalidated by specific examples (inductive) or sensory experiences (perceptual). The final proof scheme, called *transformational,* falls under the deductive category because it includes the validation of a conjecture by means of logical deductions.

Type IV: Identification or statement		Example
Continuing phenomenon: Identification of a dynamic property extending beyond a specific instance		"Every time x goes up 1, y goes up 5" or "For every second, he walks 2/3 cm."
Sameness: A statement of commonality or similarity	*Common property:* Identification of the property common to objects or situations	"For each pair, x is a third of y."
	Objects or representations: Identification of objects as similar or identical	"Even though those equations look different, they're both relating distance and time."
	Situations: Identification of situations as similar or identical	"This gear problem is just like the swimming laps problem we did in class!"
General principle: A statement of a general phenomenon	*Rule:* Description of a general formula or fact	$s \cdot (2/3) = b$, or "You multiply the number the small gear turns by 2/3 to get the number the big gear turns."
	Pattern: Identification of a general pattern	"On the x side it's going up by 1's, and on the y side it's going up by 7's."
	Strategy or procedure: Description of a method extending beyond a specific case	"To find out if each pair represents the same speed, divide miles by hours and see if the ratio is always the same."
	Global rule: Statement of the meaning of an object or idea	"If the rate of change stays the same, the data are linear."
Type V: Definition		**Example**
Class of objects: Definition of a class of objects all satisfying a given relationship, pattern, or other phenomenon		"Any two gears with a 2:3 ratio of teeth will also have a 2:3 ratio of revolutions."
Type VI: Influence		**Examples**
Prior idea or strategy: Implementation of a previously developed generalization		"You could do the same thing on this speed problem that I did with the gears. Divide y by x, and you should get the same thing."
Modified idea or strategy: Adaptation of an existing generalization to apply to a new problem or situation		"Dividing y by x each time doesn't work on this problem, but you could divide the increase in y by the increase in x instead."

Fig. 8.2. Reflection generalizations

EXTERNAL CONVICTION	Examples
Authoritarian: The main source of conviction is a statement made by a teacher or appearing in a text.	*Student*: [Looking at a table of data]. Since the y-values increase by the same number each time, this will be a straight line. *Interviewer*: Why? *Student*: That's what Ms. R told us.
Symbolic: Students view and manipulate symbols without reference to any functional or quantitative reference.	[Given three connected gears that rotate 6, 4, and 3 times respectively, students decide that another triple could be 12, 8, and 6 rotations.] *Teacher*: Why is that valid? *Student*: There's a pattern in all of them. So if you do one thing to the small one, you have to do it to the middle one and the big one to keep the ratios the same. *Teacher*: Why does that work? *Student*: It's kind of like changing fractions from 1/2 to 3/6. It's the same thing, just in different form.

Fig. 8.3. Relevant proof schemes from Harel and Sowder (1998)

EMPIRICAL	Examples
Inductive: Students demonstrate a conjecture's truth by showing that it works with a few examples.	*Student 1*: All of these pairs must be the same speed because cross-multiplying gives the same answer each time. *Teacher*: How do you know that means they're the same speed? *Student 1*: Because 27 times 5 and 7 and 1/2 times 18 equals the same thing. *Student 2*: I tried it for all the pairs in the table and it works every time.
Perceptual: One relies on perceptual observations; for instance, making judgments based on a picture.	[The student creates a graph from a table of ordered pairs: (2, 9); (5, 22.5); (12, 54); (16, 72)]. *Student*: It couldn't be a straight line. *Interviewer*: Why not? *Student*: If you made your graph, it doesn't look like it'd be a straight line. (Sketching a graph that appears curved).
DEDUCTIVE	**Example**
Transformational: "The transformational proof scheme is characterized by (a) consideration of the generality aspects of the conjecture, (b) application of mental operations that are goal oriented and anticipatory, and (c) transformations of images as part of a deductive process." (p. 261) These proofs represent appropriate deductive reasoning.	[A student explains why he thinks $(3/4)m = b$ should describe the relationship between two gears with 12 and 16 teeth.] *Student*: Since there's 3/4 of the teeth on the small one, the big one always has 1/4 teeth to make up every turn. Making it, the big one turns 3/4 of a turn every time the small one turns once. And so, say it went through 12 teeth on the small gear and 12 on the big gear. That's only 3/4 of a turn for the big gear, while it's a full turn for the small gear. The student could mentally rotate the gears in coordination matching teeth to teeth, and then multiplicatively compare the remaining teeth to the total teeth. He operated on the gears and their rotations, and could anticipate the results of those operations. While he provided a specific example, he also understood that the ratio would remain constant for any number of turns.

Fig. 8.3. ***Continued***

Method

The study was situated at a public middle school located near a large southwestern city. Seven seventh-grade pre-algebra students were selected to participate in a fifteen-day teaching experiment, in which the students met for an hour and a half each day. Every student who volunteered for the study was accepted, which resulted in a sample of six girls and one boy. One student was an English language learner. All sessions were videotaped and transcribed.

The purpose of the teaching experiment was to explore students' generalizations and justifications as they engaged in the context of realistic problems about linear growth. The sessions therefore focused on real-world situations involving gear ratios and constant speed. For the first seven days, the students worked with physical gears to explore gear ratios. For the remaining eight days, the students worked with the speed simulation computer program SimCalc Mathworlds, which students could use to generate and test conjectures about how different combinations of distance and time affected the characters' walking speed. Figure 8.4 provides a sample of the types of problems that the students encountered.

Review of the entire data set revealed major trends in the growth of students' generalizing. Early in the sessions, students focused on generalizing from immediate relationships between quantities; in the later sessions, they generalized across different quantitative situations in order to establish more global rules about linearity. In addition, students' justifications evolved over time from those that were symbolic and empirical to those that were transformational. In attempting to capture the nature of increased sophistication, students' later generalizations and justifications were contrasted to those that they had produced earlier. The four mechanisms for change emerged as a way to explain this growth in sophistication.

Connected Gears Problem

You have 2 gears on your table, one with 8 teeth and one with 12 teeth. Answer the following questions:

1. If you turn the small gear a certain number of times, does the big gear turn more revolutions, fewer, or the same amount? How can you tell?
2. Devise a way to keep track of how many revolutions the small gear makes. Devise a way to keep track of the revolutions the big gear makes. How can you keep track of both at the same time?
3. How many times will the small gear turn if the big gear turns 64 times? How many times will the big gear turn if the small gear turns 192 times?

Frog Walking Problem

The table shows some of the distances and times that Frog traveled. Is he going the same speed the whole time, or is he speeding up or slowing down? How can you tell?

Distance	Time
3.75 cm	1.5 sec
7.5 cm	3 sec
12 cm	4.8 sec
15 cm	6 sec
40 cm	16 sec

Fig. 8.4. Sample teaching-experiment problems

Results

Results from the teaching experiment suggest that relationships between generalizing and proving were rarely self-contained; students did not generalize in one way, provide a particular type of justification for that generalization, and then move on. Instead, they built up increasingly sophisticated generalizations and justifications over time, each mathematical action contributing to the evolution of the other. The four mechanisms are a way to describe how generalizing and proving can mutually influence one another to support more sophisticated reasoning. In this section, the four mechanisms are presented first, and they are then exemplified in a data episode in which students develop, explain, and justify a set of equivalent ratios.

Four mechanisms for change

The first mechanism is the *action/reflection cycle*, in which students engage in a generalizing action, formalize the action as a reflection generalization, and then shift to a new generalizing action. Although students' initial generalizations were often limited or even incorrect, subsequent cycles of generalizing built on their previous attempts to increase the sophistication of their generalizations.

The second mechanism is *mathematical focus*, which refers to how the students focused their attention (rather than the mathematical focus intended by the teacher or a particular problem). Students' focus affected both how they generalized and the types of proofs they attempted. In particular, when students focused on number patterns, they generalized by searching for patterns, extending by continuing, and made statements of principles related to patterns. Their associated justifications relied on the external symbolic and empirical proof schemes 71 percent of the time. In contrast, when students focused on relationships between quantities (measurable attributes such as length, distance, time, rotations, or speed), they generalized by searching for the same relationship and made reflection generalizations such as statements of continuing phenomena and statements of general principles related to quantities. Impressively, the associated justifications for these generalizations relied on the transformational proof scheme 67 percent of the time. These results suggest that the mathematical properties to which students attend can either inhibit or promote more sophisticated reasoning, with a quantitative focus being more effective than a focus on number patterns that are disconnected from the problem's context.

The development of the transformational proof scheme was one of the aims of the teaching experiment, because it represents a shift away from example-based reasoning to deductive argumentation. Three types of generalizations were connected to the transformational proof scheme: (*a*) the action of searching for the same relationship, (*b*) the action of extending, and (*c*) the statement of a continuing phenomenon. The connection between these three types of generalizations and the transformational proof scheme is the third mechanism of change, *generalizations promoting deductive proof*.

The fourth mechanism addresses the role the transformational proof scheme plays in *promoting more powerful generalizations*. In particular, deductive arguments promoted shifts towards two types of reflection generalizations: (*a*) statements of continuing phenomena, and (*b*) new general principles such as rules, patterns, and global rules. This finding suggests that students can begin with generalizations that may be limited, but after justifying with the transformational proof scheme, they may subsequently develop more accurate, sophisticated generalizations.

Data episode: Explaining equivalent ratios

The lesson in this episode began with the following problem: "Say Clown walks 15 cm in 12 seconds. Find as many different ways as you can to make Frog walk the same speed as Clown." Students developed a number of same-speed pairs by operating on a 5 cm:4 s unit, and concluded that any multiple of 5 cm in 4 s would represent the same speed. Then one student, Timothy, realized that there were other same-speed pairs beyond multiples of 5 cm in 4 s. He explained, "Oh! You multiply whatever the centimeters are by 4/5 to equal the seconds. So it doesn't matter if they're multiples [of 5 cm:4 s]." Timothy's realization was a reflection generalization, the identification of a general rule, but none of the students could explain why it worked:

Teacher: Now I have not heard a justification why this works.

Julie: Because the numbers 12 and 15.

Timothy: 12 over 15 equals 4/5.

Dani: You can simply and that's why. You can simplify 15 over 12.

Julie: You can simplify it down.

Teacher: So those are all forms of 12 and 15?

Julie: Yeah. It has to be a form of 12 and 15.

Teacher: Why?

Timothy: I don't know. It just works!

The following day began with a follow-up requesting students to justify their reasoning by drawing a picture. Timothy drew a graph (see fig. 8.5), which he viewed as a way to show that both Clown and Frog walked the same rate. The teacher asked Timothy to think about what was the same across two points in his graph, (2.5, 2) and (20, 16), and Timothy explained, “They’re all the same centimeters per second. The amount of centimeters that Frog’s gonna run in a certain amount of time.” The teacher’s question was designed to direct Timothy’s attention to sameness across the points, which likely afforded his generalizing action of searching for the same relationship because he had to think about what relationship held between (2.5, 2) and (20, 16). This led to Timothy’s reflection generalization, which was the identification of the property common to all the points. Timothy was then able to identify the line’s slope as 4/5.

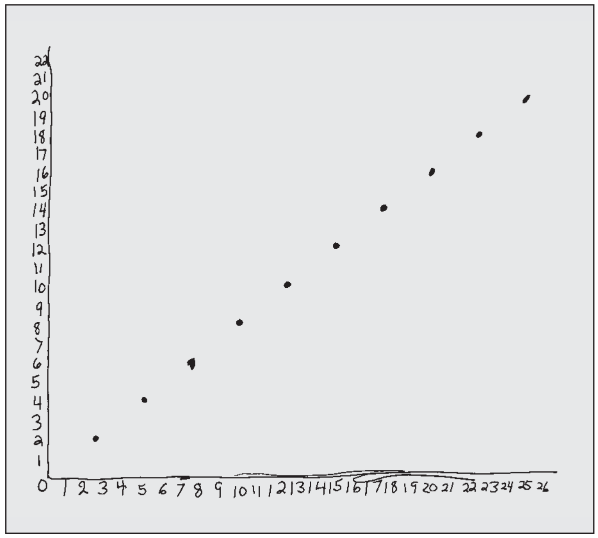

Fig. 8.5. Timothy’s graph showing same-speed points

Timothy: So that means, since the slope is 4/5, this [gesturing to the *y* column] is 4/5 of this [gesturing to the *x* column]. Basically, whatever *y* is, is 4/5 of whatever *x* is.

Teacher: Oh, and how does that 4/5 slope relate to what you’re figuring out with speed?

Timothy: Because let’s see . . . for every centimeter it goes, it’s going like, 4, er, yeah 4/5 of a second I think . . . So for 15 cm, 4/5 of 15 would equal 12. So since there’s 15 cm, 4/5 of 15 cm, for every centimeter it’s going 4/5 of a second, or 12 s.

Timothy’s statement “whatever *y* is, is 4/5 of whatever *x* is” represents an identification of a general pattern. When asked to connect the pattern to the phenomenon of speed, he produced a different generalization. Timothy identified a continuing phenomenon, pointing to the dynamic relationship between centimeters and seconds. In addition, he made a connection between 4/5 and same speed; the 4/5 pattern now had meaning in terms of the quantities in the situation. When the group convened, Timothy shared his justification with the group, explaining:

> The graph’s showing the different amount of centimeters and different amount of time they could have taken. And since, as long as you did like one of these [gesturing to the line] or beyond that, you would always end up having them go along at the same speed. Because since 15 and 12 is also on that line. And so . . . since 15 and 12 is also on that line, and you do anything else that’s on that line, you’ll be going at the same speed. Just one of them will stop at a certain time.

Timothy’s reasoning had some elements of the transformational proof scheme; he could imagine any given point as representing the given speed. In addition, Timothy's statement that "you do anything else that's on the line" suggests that he anticipated that any point on the line, not just the ones drawn from his table, would represent the same speed. Moreover, Timothy explained to his classmates that "the slope means that whatever *x* goes up by . . . 4/5 of that is how much *y* goes up by." This marks an important difference from Timothy's prior statement. He is no longer stating that *y* is 4/5 of *x*; instead, he now understands that any *increase* in *y* will be 4/5 of the same increase in *x*. This is a statement of a general pattern, as before, but it is a more powerful general pattern; it applies to all linear functions of the form $y = mx + b$ rather than just functions of the form $y = mx$.

Meanwhile, another pair of students, Larissa and Maria, produced a drawing rather than a graph (see fig. 8.6). The top number line shows Frog's journey and the bottom shows Clown's journey, with the boxed numbers representing the number of seconds corresponding to the number of centimeters on each number line. For example, the boxed 4.0 directly above and below the 5 on each line shows that at 5 cm, both Frog and Clown had traveled for 4 seconds.

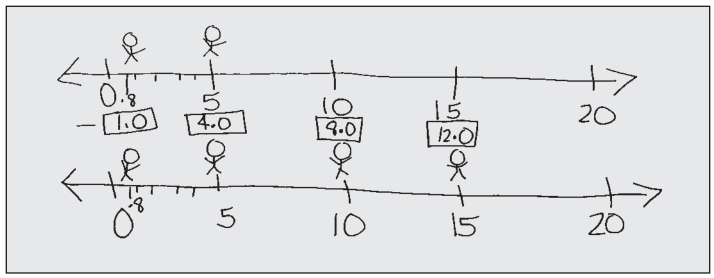

Fig. 8.6. Larissa and Maria's diagram of Clown and Frog walking

The students explain how their picture shows that Frog and Clown walked the same speed:

Maria: Okay, we figured out that every 8, 0.8 s, no, every second you go 0.8 cm.

Timothy: I think it's the other way around.

Dora: Every centimeter you go 0.8 s.

Timothy: Because that would explain the 15 cm, 12 s. Because the smaller amount of seconds.

Maria: Okay. And in the 4 s the Frog reached 5 cm, and that was the speed of the Clown. In 12 s, 15 cm. In 8 s, the 10 cm. In 4 s, he reached the 5 cm.

Teacher: Excellent. Now Larissa, can you explain how this picture shows that Frog and Clown are going the same speed?

Larissa: Because for the, when they're at 4, both of them are at 4 s. But since the frog stops, he's finished . . . but the clown keeps going and from 0 to 5 it jumped 4 s . . . and from 5 to 10 it also jumped 5 cm and 4 s. And from 10 to 15, it jumped 5 cm and also 4 seconds. So the proportion stays the same throughout the whole thing even though Frog stopped.

Maria produced the reflection generalization of an identification of a continuing phenomenon: For every 1 cm the clown walked, it took 0.8 s. (Although she stated it incorrectly, Maria's written work showed the correct relationship, suggesting she misspoke). So Maria made a new statement about the speed situation, one that the students had not previously realized. Furthermore, Larissa's justification revealed elements of the transformational proof scheme, because she could imagine the Frog completing his journey at the same proportion as the Clown's completed journey. When subsequently asked if she could generalize her argument to any two same-speed characters, Larissa said, "If the two objects are walking the same speed, then the proportion throughout . . . their walking will stay the same even if one of the objects stops." Through discussing the picture and her justification, Larissa could now state a more general idea. She engaged in the generalizing action of extending by removing the particulars, because she had extended the idea of keeping the same proportion to any same speed pair. Larissa had also produced the reflection generalization that if the speed is the same, the proportion will remain the same: a global rule.

Mechanisms for change in practice

The following sections describe how the four mechanisms for change occurred in students' activity.

Mechanism 1: Action/Reflection

On the prior day, Timothy had identified a general rule: multiply the centimeters by 4/5 to get the seconds. By creating a graph and focusing on sameness across the points, Timothy's generalizing action of searching for the same relationship led to another reflection generalization—the identification of a general pattern: "Whatever y is, is 4/5 of whatever x is." Once he tried to connect this pattern to speed, Timothy realized that for every centimeter the Frog walked, it took 4/5 of a second. This reflection generalization is different from the others. First, it is an identification of a continuing phenomenon, and so it is stated in a different form. But more importantly, it was the first time that one of the students stated a connection between the general pattern and the speed situation. The 4/5 now carried a quantitative meaning for Timothy. Another cycle of generalizing and justifying produced Timothy's final reflection generalization, the statement of a different general pattern conveying the meaning of slope. This is arguably more sophisticated than the generalizations that preceded it, because now Timothy understood that not only are seconds proportional to centimeters, but the *change* in seconds is proportional to the *change* in centimeters.

Meanwhile, Maria also produced a statement of a continuing phenomenon, and this generalization also made sense of the speed situation whereas the girls' prior generalizations did not. It is possible that the girls' generalizing action of extending, which occurred when they repeated and partitioned the 5 cm:4 s unit to produce their diagram, helped solidify the continuing phenomenon generalization. Subsequently, when asked to generalize to any same-speed situation, Larissa engaged in extending by removing particulars, which resulted in the identification of a global rule. Larissa has now developed an inference about the meaning of same speed, and her understanding is not restricted to one particular problem.

Mechanism 2: Focus

The episode shows how a focus on quantities can result in generalizations that connect number patterns to situations, as well as in justifications with the transformational proof scheme. Before the students made a strong connection to speed on the second day, their attempts to justify why multiples of 4/5 resulted in same-speed values were dependent on attempts to manipulate symbols they did not fully understand, and thus were limited to the symbolic proof scheme. Once the students focused on the quantities of centimeters and seconds, they began to connect their patterns to the speed phenomenon. This is seen in Timothy's description of the slope as a representation of speed, and in Maria and Larissa's statement that the clown took 0.8 s to walk 1 cm. Furthermore, attending to the relationships between centimeters and seconds helped the students develop more powerful justifications and make new inferences about the problem.

Mechanism 3: Generalizations promoting deductive proof

The students' generalizing actions of searching (as Timothy searched for the same relationship across points) and extending (as Larissa and Maria expanded the 5 cm:4 s ratio into a more general structure) appeared to strengthen their understanding of the invariant multiplicative relationship between centimeters and seconds. All of the students also identified continuing phenomena, which supported a focus on a more general dynamic relationship between quantities and the identification of a property that extends through time. These three types of generalizing were the ones most closely tied to the use of the transformational proof scheme.

Mechanism 4: Influence of deductive reasoning on generalizing

Both Timothy's justification and Larissa and Maria's justification enabled the identification of new general principles, patterns, and global rules. Timothy's explanation of how his graph represented same speed values focused his attention on the ratio of increases, which resulted in a new generalization. Larissa's act of justifying helped her generalize further by extending her reasoning to any same-speed pair. In general, there were two major types of generalizations that appeared after students produced deductive justifications: identification of general principles, such as algebraic or global rules, and identifications of continuing phenomena. These episodes suggest that the relationship between generalizing and justifying is not unidirectional. Students do not produce a generalization, justify it, and then move on. Instead, the act of justifying itself can push students' reasoning forward in ways that encourage further generalizing.

Conclusion and Implications for Teaching

A critical instructional goal is to help students develop correct algebraic generalizations and deductive forms of proof. This study suggests that while it is important not to ignore these goals, students' incorrect or partially correct generalizations and unsophisticated proof attempts can serve as an important bridge towards achieving them. Given the growing emphasis on proof in the middle grades, it could be helpful to better understand which types of generalizing activities can support more powerful justifications, particularly in guiding lesson development. In order to encourage more deductive argumentation, teachers could design problem situations that encourage students to (*a*) search for similar relationships across problems, cases, or contexts; (*b*) extend their reasoning about patterns, relationships, and phenomena; and (*c*) develop statements about the quantitative relationships they see in the problem context.

Results from the study also address the role of justifying as a support for better generalizing. Recent pedagogical recommendations encourage teachers to present tasks in which algebra students find and generalize patterns (NCTM, 2000). These recommendations reflect an assumption that generalizing patterns is sufficient support for producing appropriate proofs, but research demonstrates that it is not (Knuth & Elliott, 1998; Lannin, 2005). A more productive approach to proof instruction may challenge the typical generalization/proof sequence. Students in the study initially engaged in generalizing activities that were at times limited, partially incorrect, or otherwise unproductive. However, as they attempted to explain their generalizations and create increasingly deductive justifications, students were able to revisit their generalizing actions, build on them, and ultimately construct ones that were more powerful. The students' engagement in increasingly sophisticated generalization/justification cycles suggests that teachers might consider incorporating justifying early into the instructional sequence, rather than expecting students to produce their final generalizations before moving on to proof. The role of proof could therefore be viewed as a way to help students generalize more effectively, rather than as an act that necessarily follows generalization.

In order to facilitate early engagement with proof, teachers should emphasize problems that allow for appropriate justification. In the context of linear functions, this would mean de-emphasizing situations in which the data are contrived or inexact in favor of situations with data that students can investigate, manipulate, and make sense of. Moreover, problem situations encouraging a focus on relationships between quantities instead of number patterns or procedures alone could provide a more fruitful setting to encourage productive generalizing and justifying. Teachers also play an important role in helping their students focus attention on quantities and the language of quantitative relationships. Although students often attend to number patterns alone, even within the context of a quantitatively rich problem, teachers can intervene to draw students' attention back toward the quantitative referents, and to incorporate the language of quantities into the classroom discussion by asking students to shift from pattern descriptions to phenomenon descriptions.

The four mechanisms of change show that students can, and do, move from less productive actions to more powerful generalizing and justifying behaviors over time. Student strategies or solutions that may appear to be unacceptable due to their limited nature could be the very ones that ultimately support more powerful ideas over time.

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