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The Influence of Reasoning with Emergent Quantities on Students' Generalizations

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This paper reports the mathematical generalizations of two groups of algebra students, one which focused primarily on quantitative relationships, and one which focused primarily on number patterns disconnected from quantities. Results indicate that instruction encouraging a focus on number patterns supported generalizations about patterns, procedures, and rules, while instruction encouraging a focus on quantities supported generalizations about relationships, connections between situations, and dynamic phenomena, such as the nature of constant speed. An examination of the similarities and differences in students' generalizations revealed that the type of quantitative reasoning in which students engaged ultimately proved more important in influencing their generalizing than a mere focus on quantities versus numbers. In order to develop powerful, global generalizations about relationships, students had to construct ratios as emergent quantities relating two initial quantities. The role of emergent-ratio quantities is discussed as it relates to pedagogical practices that can support students' abilities to correctly generalize.

Expanded Notions of Algebra: The Importance of Quantitative Reasoning

Approaches to teaching algebra have recently expanded to include an emphasis on developing students' ability to create powerful generalizations. For instance, the National Research Council notes that in addition to symbol manipulation, school algebra should also include representational activities and generalizing and justifying activities (Kilpatrick, Swafford, & Findell, 2001). This shift in emphasis is one

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response to research showing that traditional algebra courses focusing on strategies for symbol manipulation, simplifying expressions, and solving equations yield poor results in overcoming the well-documented difficulties students experience in understanding algebra (Booth, 1988; Kieran, 1992; Stacey & MacGregor, 1997). Expanded notions of what should constitute school algebra have since become a focus of research and discussion (Kaput, 1998; NCTM, 2000; van Reeuwijk, in press; Romberg, 1998). While researchers' views overlap and differ in their emphases, there is general agreement that algebraic activity should move beyond symbol manipulation to also focus on reasoning with patterns, quantities, and real-world situations. These views are reflected in the development of curricula that promote such activities in the algebra classroom (Coxford, Fey, Schoen, Burrill, Hart, Watkins et al., 1998; Lappan, Fey, Fitzgerald, Friel, & Phillips, 1998).

One alternative approach is based on the introduction of algebraic ideas through an exploration of quantities and quantitative relationships (Fujii & Stephens, 2001; Kaput, 1995; Steffe & Izsak, 2002). Thompson (1994) defined a quantity as one's conception of a measurable quality of an object or a phenomenon. It is composed of an object or event, a quality of the object, an appropriate unit or dimension, and a process for assigning a numerical value to the quality. Length, area, volume, cardinality, speed, temperature, and density are all attributes that can be measured in quantities. Quantities are constituted in people's conceptions of situations, rather than the objects or situations themselves.

While this definition may seem simple, it assumes that the student has isolated an object and a relevant attribute to be measured (Thompson, 1990). This condition is often not as obvious to children as it is to their teachers. Hall and Rubin (1998) caution "it is a central conceptual problem for many students to distinguish between different types of quantity . . . and to find strategies that can generate one of these quantities from another two" (p. 199). Studies from the modeling literature demonstrate students' difficulties in establishing quantities and finding ways to measure them; for instance, Lehrer and Schauble (2004) found that when students measured heights of individual plants, they could not readily determine either the attribute they were supposed to focus on or its measure. Students may focus on only one salient object, ignoring other relevant quantities (Lobato & Thanheiser, 2002). These difficulties are likely tied to a lack of opportunity to regularly participate in contexts that allow students to consider qualities of measures and attributes relevant to mathematical questions (Petrosino, 2003; Petrosino, Lehrer, & Schauble, 2003). In response, Petrosino (2003) advocates providing students with opportunities to discover which properties of measure need to be emphasized in measuring an attribute. These experiences can help children understand that measure incorporates ideas such as iteration, origin, and equal units.

The quantitative reasoning approach emphasizes operating with quantities and their relationships. A quantitative operation is a conceptual operation by which one conceives a new quantity in relation to one or more already-conceived quantities.

Quantitative operations are nonnumeric and emerge from one's evaluation of a situation, while arithmetic operations are used to evaluate quantities (Lobato & Siebert, 2002). For example, one might compare quantities additively, by comparing how much taller one person is to another, or multiplicatively, by asking how many times bigger one object is than another. The associated arithmetic operations would be subtraction and division. Although researchers may occasionally blur the distinction between reasoning quantitatively and reasoning with real-world situations, the two are not identical. The manner in which a student interacts with a given situation determines whether he or she is reasoning quantitatively rather than the nature of the situation itself. Thus a student could attend to number patterns extracted from a real-world situation and be engaged in number pattern reasoning alone. Similarly, a student could examine relationships between quantities in a highly unrealistic, abstract, or imaginary situation and still be engaged in quantitative reasoning.

Thompson (1994b, April) suggests that students develop mathematical conceptions through quantitative reasoning rather than through the study of algorithms or procedures. Quantitative operations originate in actions, or activities of the mind (Piaget, 1967). As a learner interiorizes actions, creating mental operations, these operations allow one to comprehend situations representationally. They enable the learner to draw inferences, for example, about relationships that may not be present in the situation itself. If all mental actions are tied to experience, then any meaningful learning in mathematics must be grounded in quantitative referents. As Steffe and Izsák (2002) remark, "We regard quantitative reasoning as the basis for algebraic reasoning" (p. 1164).

This approach towards algebraic reasoning marks a departure from earlier conceptions that characterize mathematics as the study of patterns and relationships, regardless of their origin (AAAS, 1993). In Thompson's (1994b) words, "This view is problematic because it suggests that numbers, shape, and relationships are given, that they are the primary starting point for mathematical inquiry" (pp. 7–8). Rather than asking students to study objects that have not emerged from their own mental activity, researchers are beginning to emphasize the need to build from students' existing mental operations, which could be achieved by focusing on quantities in realistic situations. For this reason, quantitative reasoning has become more closely tied to what it means to reason algebraically (Fujii & Stephens, 2001; Kaput, 1995; Steffe & Izsák, 2002).

Quantitative Reasoning and Generalization

Studies investigating algebra students' generalizations with number patterns suggest that students experience difficulty recognizing and forming correct general statements (English &Warren, 1995; Kieran, 1992; Lee & Wheeler, 1987). Although students can recognize multiple patterns, they may not distinguish those which are algebraically useful. In addition, the perception of a valid number pattern does not mean that students will generalize that pattern correctly (Blanton & Kaput, 2002, April; Ellis, 2007a; English & Warren, 1995; Lee, 1996; Lee &Wheeler, 1987; Orton & Orton, 1994; Stacey, 1989; Stacey & MacGregor, 1997). In particular, students tend to focus on recursive relationships, describing the nth term in relationship to the preceding term(s), which can impede their ability to create explicit algebraic generalizations (Pegg & Redden, 1990; Schliemann, Carraher, & Brizuela, 2001; Szombathelyi & Szarvas, 1998).

In contrast, some studies exploring how students reason with algebraic relationships in quantitatively-rich situations have shown that students can notice patterns, identify relationships, and create generalizations that all hold quantitative meaning for them (Curcio, Nimerofsky, Perez, & Yaloz, 1997; Ellis, 2007b; Hall & Rubin, 1998; Lobato & Siebert, 2002; Noble, Nemirovsky, Wright, & Tierney, 2001; van Reeuwijk & Wijers, 1997; Thompson, 1994). For instance, in Lobato and Siebert's (2002) study, students participated in a teaching experiment focused on constant speed situations. The participants initially developed patterns involving doubling and tripling the distance and time one character traveled to produce same speed values, which they eventually generalized into an argument that one could take different multiples of the character's distance and time values without changing his speed. One student was then able to form a composite "10 cm in 4 s" unit (Lamon, 1995), which was adopted by the other students. They then further generalized by partitioning and iterating the composite unit to determine that the character could repeat his journey any number of times, or could travel a fraction of his journey, without changing his speed. The students' reasoning was strongly connected to the speed situation; they attended to the relevant quantities while discussing their ideas, and their generalizations frequently referenced characters walking a certain number of centimeters in a certain number of seconds.

Studies focused on models and modeling similarly report students' abilities to engage in multiple iterations of model development in order to create emergent quantities such as ratios and proportions (Lesh & Harel, 2003; Lesh & Lehrer, 2003). For instance, Sherin (2000) describes students' representations of the emergent quantity of varying types of speed. Students were charged with the task of developing models of different types of motion described verbally, and Sherin conveyed the students' development, adaptation, extension, and evolution of various representations of motion. By developing representations grounded in their understanding of emergent quantities, students were able to create meaningful generalizations. Lehrer, Schauble, Strom, and Pligge (2001) reported on a study in which students developed an understanding of density as a constant ratio. The students were able to build on their ratio reasoning to determine which types of solids and liquids would float or sink in water. In another study, Lehrer, Schauble, Carpenter, and Penner (2000) described students' focus on changing ratios in order to investigate plant growth. In each of these cases, students' experientially real world is used as a meaningful base for the development of mathematical concepts and skills.

Kaput (1999) anticipated the type of student understanding described above when he noted that “students are more likely to begin by generalizing from their conceptions of situations experienced as meaningful and to derive their formalizations from conceptual activities based in those situations” (p. 137). However, merely placing students in quantitatively-rich situations and asking them to generalize will not necessarily result in students noticing relationships that are correct, helpful, or even meaningful to them. While the above examples from the literature make the case that quantitatively meaningful generalizations are possible, other studies demonstrate that these generalizations do not always occur. Much of the literature related to modeling describes students’ difficulties in making sense of realistic, messy data (Metz, 2004; Petrosino, 2003) and modeling quantities that are emergent rather than explicit (Lehrer & Schauble, 2004). Metz (2004) noted that students tend not to think their findings will generalize when they develop their own data, as they view their knowledge as mediated by their study and its limitations.

In one example, Noble et al. (2001) described a case in which the initial patterns developed by the students were not helpful to their attempts to extend their reasoning. The students developed number tables in which they noticed multiple patterns, but initially struggled to create algebraically useful generalizations. The authors noted that the patterns the students noticed were such that were such that one pattern did not seem more significant or useful than another. van Reeuwijk and Wijer (1997) reported a similar phenomenon; although students eventually developed generalizations closely tied to relevant quantities, their initial perceptions were more numeric. In each of these examples, students focused on number patterns that did not carry quantitative meaning in part because they reasoned with artifacts, such as number tables, that were disconnected from situational attributes. Later, when they worked more closely with their conceptions of the quantitative relationships through other mechanisms such as pictures, diagrams, or physical portrayals, their generalizations were more quantitative than numeric.

This paper reports the mathematical generalizations of two groups of students, one which focused primarily on quantitative relationships, and one which focused primarily on number patterns disconnected from quantities. The types of generalizations the students produced differed according to their focus: those who attended to number patterns created generalizations about patterns and rules, while those who attended to quantities created generalizations about relationships and dynamic phenomena such as gear relationships and speed. Results indicated, however, that a focus on quantities was ultimately less important than the *type* of quantitative reasoning in which students engaged. The differences in the students’ generalizations are reported, followed by a discussion of the importance of a particular type of “emergent-ratio quantitative reasoning” that relies on forming emergent quantities from ratios of initial quantities.

The Generalization Taxonomy

Ellis’s (2007a) generalization taxonomy describes the different types of generalizations students develop when reasoning algebraically. In the spirit of Kaput’s (1999) view, generalization is defined as engaging in at least one of three activities: a) identifying commonality across cases, b) extending one’s reasoning beyond the range in which it originated, or c) deriving broader results about new relationships from particular cases. In each case, a student’s reasoning is not limited to the confines of one particular problem, situation, or set of numbers. Instead, one extends an idea to a wider range of phenomena, and in doing so, generalizes that idea; Piaget & Henriques (1978) called this generalizing assimilation. While the notions of extending one’s reasoning and deriving broad results can be easily tied to generalizing assimilation, the connection may appear less obvious for the first criterion, identifying commonality across cases. However, in identifying a common element across cases, one must consider that element as it relates to more than one specific case. One must either incorporate more than a single instance of a phenomenon into a concept, or be able to locate evidence of the concept in more than one instance. In either case, this perspective requires broadening one’s notion of the concept itself. In identifying commonality, one therefore considers the idea in question in a broader context than the particular case in which it arose, which is another form of extending an idea to a wider range of phenomena.

Evidence for generalization is not pre-determined, but instead relies on identifying the similarities and extensions that students perceive as general. This view of generalization is more expansive than the typical approach in which a formal verbal or algebraic description of a correct rule is required as evidence of generalization (Orton & Orton, 1994; Stacey & MacGregor, 1997). This wider definition allows researchers to capture a greater number of student acts as generalizing.

Figures 1 and 2 outline the types of generalizations identified in the taxonomy (for full descriptions, see Ellis, 2007a). The taxonomy distinguishes between students’ mental activity as they generalize, called *generalizing actions*, and students’ final statements of generalization, called *reflection generalizations*. Generalizing actions describe learners’ mental and mathematical actions—they are mental acts that are inferred through a person’s activity and talk. An examination of problem-solving behavior, such as the mathematical operations a student employs while working with a problem, a student’s apparent mathematical focus, the properties and relationships a student attends to, or the strategies in which a student engages, can lead to a description of the types of mental actions the student appears to employ in his or her attempts to generalize. The behaviors themselves do not constitute the generalizing actions, but contribute to the researcher’s determination

<table>
<tr><th colspan="2">TYPE I: RELATING</th><th>Examples</th></tr>
<tr><td rowspan="2">Relating Situations: The formation of an association between two or more problems or situations.</td><td>Connecting Back: Connecting between a current and previously-encountered situation.</td><td>Realizing that “This gear problem is just like the swimming laps problem we did in class!”</td></tr>
<tr><td>Creating New: Inventing a new situation viewed as similar to an existing one.</td><td>“He’s walking 5 cm every 2 s. It’d be like a heart that was beating at a steady pace, 5 beats in 2 s.”</td></tr>
<tr><td rowspan="2">Relating Objects: The formation of an association of similarity between two or more present objects.</td><td>Property: Associating objects by focusing on a property similar to both.</td><td>Noticing that two equations in different forms both show a multiplicative relationship between x and y.</td></tr>
<tr><td>Form: Associating objects by focusing on their similar form.</td><td>Noticing that “Those equations both have one thing divided by another.”</td></tr>
<tr><th colspan="2">TYPE II: SEARCHING</th><th>Examples</th></tr>
<tr><td colspan="2">Same Relationship: Performing a repeated action in order to detect a stable relationship between two or more objects.</td><td>Dividing y by x for each ordered pair in a distance/time table to determine if the speed remains the same.</td></tr>
<tr><td colspan="2">Same Procedure: Repeatedly performing a procedure in order to test whether it remains valid for all cases.*

*A searching action is coded as a relationship or a procedure based on the researcher’s understanding of the student’s understanding. One can perform the same calculational action in both cases, but the meaning of that action for the student determines whether she is searching for a relationship or performing a procedure.</td><td>Dividing y by x as above without understanding what quantitative relationship is revealed by division; dividing as an arithmetic procedure to determine whether the resulting answer is the same.</td></tr>
<tr><td colspan="2">Same Pattern: Checking whether a detected pattern remains stable across all cases.</td><td>Given a series of connected matchstick triangles, noting that each extra triangle requires two more matchsticks.</td></tr>
<tr><td colspan="2">Same Solution or Result: Performing a repeated action in order to determine if the outcome of the action is identical every time.</td><td>Given an equation such as $y = 2x$, substituting multiple integers for x and noticing that y is always even.</td></tr>
<tr><th colspan="2">TYPE III: EXTENDNG</th><th>Examples</th></tr>
<tr><td colspan="2">Expanding the Range of Applicability: Applying a phenomenon to a larger range of cases than that from which it originated.</td><td>Having graphed discreet points representing constant speed, noting that it would be possible to extend the graph indefinitely for positive x- and y-values.</td></tr>
<tr><td colspan="2">Removing Particulars: Removing some contextual details in order to develop a global case.</td><td>Having identified that two people walking the same speed have the same distance/time ratio at several points on the journey, generalizing that any same-speed objects will always have the same distance/time ratio for any given location.</td></tr>
<tr><td colspan="2">Operating: Mathematically operating upon an object in order to generate new cases.</td><td>Knowing that y increases by 6 cm for every 1 s increase for x, halving the (1:6) ratio to create a new ordered pair that represents the same speed.</td></tr>
<tr><td colspan="2">Continuing: Repeating an existing pattern in order to generate new cases.</td><td>Knowing that y increases by 6 cm for every 1 s increase for x, continuing the (1:6) ratio to create new ordered pairs that represent the same speed.</td></tr>
</table>

FIGURE 1 Generalizing actions.

IDENTIFICATION OR STATEMENT		Example
Continuing Phenomenon: Identification of a dynamic property extending beyond a specific instance.		"Every time *x* goes up 1, *y* goes up 5." Or, "For every second, he walks 2/3 cm."
Sameness: A statement of commonality or similarity.	*Common Property*: Identification of the property common to objects or situations.	"For each pair in the table, the ratio of centimeters to seconds is the same. So each pair must be the same speed."
	Objects or Representations: Identification of objects as similar or identical.	"Even though those equations look different, they're both relating distance and time."
	Situations: Identification of situations as similar or identical.	"This gear problem is just like the swimming laps problem we did in class!"
General Principle: A statement of a general phenomenon.	*Rule*: Description of a general formula or fact.	$s \cdot (2/3) = b$, or "You multiply the number the small gear turns by 2/3 to get the number the big gear turns."
	Pattern: Identification of a general pattern.	"Each triangle train is the same because you can add two matchsticks every time."
	Strategy or Procedure: Description of a method extending beyond a specific case.	"To find out if each pair represents the same speed, divide miles by hours and see if the ratio is always the same."
	Global Rule: Statement of the meaning of an object or idea.	"If the rate of change stays the same, the data are linear."
DEFINITION		**Example**
Class of Objects: Definition of a class of objects all satisfying a given relationship, pattern, or other phenomenon.		"Any two gears with a 2:3 ratio of teeth will also have a 2:3 ratio of revolutions."
INFLUENCE		**Examples**
Prior Idea or Strategy: Implementation of a previously-developed generalization.		"You could do the same thing on this speed problem that I did with the gears. Look at the ratio each time and you see that they're the same speed."
Modified Idea or Strategy: Adaptation of an existing generalization to apply to a new problem or situation.		"Looking at the ratio each time doesn't work on this problem, but you could divide the increase in centimeters by the increase in seconds instead, and you see that he's walking the same speed."

FIGURE 2 Reflection generalizations.

of the type of generalizing actions in which the student may be engaged. The inferred mental acts are characterized as generalizing actions in part to distinguish them from the verbal or written expressions that comprise students' reflection generalizations.

Students' generalizing actions are characterized in three major categories: *relating*, *searching*, and *extending*. When relating, one forms an association between two or more problems, situations, ideas, or mathematical objects. Relating can include recalling a prior situation, inventing a new one, or focusing on similar properties or forms of mathematical objects. When searching, one engages in a repeated mathematical action, such as calculating a ratio or locating a pattern, in order to locate an element of similarity. Students may focus on relationships, procedures, patterns, or solutions when searching. Finally, the act of extending involves expanding a pattern, relationship, or rule into a more general structure. Students who extend widen their reasoning beyond the problem, situation, or case in which it originated.

Reflection generalizations occur in the form of a verbal or written statement, or in the use of a previously developed generalization in a new problem. Statements of generalization can take the form of *identifications or statements* of general patterns, properties, rules, or common elements, or *definitions* of classes of objects. The implementation of a prior generalization in a new problem or context is categorized as *influence*, either of a prior idea or strategy or a modified idea or strategy.

Many reflection generalizations mirror generalizing actions. For instance, statements of sameness may accompany any of the generalizing actions of relating, and statements of general principles may accompany searching actions. The actions of noticing similarity, searching for similarity, or extending reasoning can result in declarations of sameness, articulations of rules and principles, or definitions of classes. The declarations are the reflection generalizations, and the mental actions that lead to them are the generalizing actions.

Methods

The study consisted of two parts. Part A consisted of a classroom observation and interview study, and Part B consisted of a teaching experiment, both situated at a public middle school in a large southwestern city. The school has an ethnically diverse student population: out of its 1,000 students, approximately 40.8% are Hispanic, 28.2% are Caucasian, 16.7% are Filipino, 6.9% are African American, 6.3% are Asian American, 0.7% are Pacific Islander, and 0.4% are American Indian. Approximately 15% of the students are English language learners.

Part A: Classroom Study

The classroom study participants consisted of 34 eighth-grade algebra. Seven members of the class were also recruited for individual interviews. The students were recruited with the participating teacher's help on the basis of their good class attendance, their ability to verbalize their thought processes, and grades of C or higher. Those who had average-to-high grades and could articulate their thinking were desirable as interview participants because they were likely to have developed powerful generalizations in class that they could subsequently explain and discuss in the interview. All 7 students were relatively strong proportional reasoners as evidenced by their ability to successfully negotiate a series of proportional reasoning tasks. Three of the students were males and 4 were females. Four of the students were Hispanic, 2 were Caucasian, and 1 was Asian American. Two of the students were English language learners while 5 were native English speakers. Gender-preserving pseudonyms were used for all participants.

Classroom observations occurred during twelve consecutive class sessions devoted to linear functions. Each session lasted 1 h and 50 min, and ordinary classroom lessons were videotaped and transcribed. During the linear functions unit, the participating teacher relied on the use of several activities from the Connected Mathematics Project (CMP) (Lappan et al., 1998). CMP is a problem-centered curriculum that embeds mathematical ideas in realistic and engaging problems designed to help students develop both understanding and skill. The official text for the course was a more traditional Algebra I book (Larson, Boswell, Kanold, & Stiff, 2001), upon which the teacher relied for in-class practice problems and homework. The participating teacher employed a combination of real-world situations and problems in which equations or patterns did not refer to a specific context. The students sometimes worked individually in a whole-class environment, and they also participated in small-group activities at least once during each class period. Students engaged in activities such as collecting empirical data, building mathematical models for collected data, developing their own graphs and tables to describe phenomena, and practicing skills related to graphing points, solving equations, and simplifying algebraic expressions. Figure 3 provides an overview of the activities students engaged in and the mathematical ideas addressed as they were explored in the classroom during the observed unit.

Although the cooperating teacher relied on small-group activities tied to real-world problems, her instructional focus with the entire class was calculationally oriented. She emphasized procedures and skills related to solving equations and simplifying algebraic expressions, and led the classroom discussion in a way that relied on direct calculational questions and responses. For that reason, students had few opportunities to make observable generalization in the classroom setting, and over 80% of their generalizations therefore occurred in the individual interviews.

The 7 interview students each participated in one 60-min individual interview, which was videotaped and transcribed. The goal of the semi-structured interviews (Bernard, 1988) was to determine what sense the students made of the generalizations they developed, what types of explanations students provided for them, and what types of extensions and limitations students saw for their own generalizations. Thus, the model for the interviews involved taking some of the general statements students had developed in class and devising task questions addressing them. Based on the student's work, the tasks and the interviewer's questions were varied, but each task required the student to extend his or her reasoning to a larger set of cases or numbers. Figure 4 includes sample interview items posed to students.

Day	Situation	Mathematical Topics	Class Activities
1	Fahrenheit and Celsius	Meaning of equations and graphing data points	Graph points on a coordinate plane
2	None (diagnostic test)	Simplifying, solving, and evaluating equations	Diagnostic Test
3	Video rentals	Differences between $y = ax$ versus $y = bx + c$	Explore whether \$8/video for 5 days is cheaper than \$3/video for 5 days with a \$50/year membership
4	Strength of Bridges	Writing equations to describe directly-proportional situations	Collect data comparing layers of a bridge and number of weights held; write $y = ax$ equations
5	Strength of Bridges	Connections between $y = ax$ and linear graphs	Make scatterplots from data; Draw line of best fit
6	Speed of Gazelle and Cheetah	Connections between tables and linear equations	Write $y = ax$ equations and $y = ax + b$ equations from well-ordered tables
7	Number Patterns in Tables	Connections between tables and graphs	Graph scatterplots from tables; write $y = ax + b$ equations from tables
8	Swimming Speed of a Whale	Using graphs to predict values	Practice graphing and solving $y = ax + b$ equations
9	Number Patterns in Tables	Slope as "increase for y / increase for x"	Determine slope values from $y = ax$ and $y = ax + b$ tables
10	Betty's Bakery	Slope in tables, equations, and graphs	Write equations from tables; draw graphs from equations and tables

FIGURE 3 Overview of the linear functions unit.

Part B: Teaching Experiment

Seven seventh-grade pre-algebra students were selected for the teaching experiment on the basis of the same criteria used for the interview participants, but the recruitment population was seventh-graders rather than eighth-graders in order to select participants who had not yet received instruction on linear functions. Only 7 students from the recruitment pool of 70 volunteered for the course, and every student who volunteered was accepted. Six students were female and 1 was male. Like the interview students, the teaching-experiment students were strong proportional reasoners as evidenced by their ability to successfully negotiate a series of proportional reasoning tasks. Three students were Hispanic, 3 were Caucasian, and 1 was Asian-American. One student was an English language learner, and the other 6 were native English speakers.

All teaching-experiment sessions were taught by the author and were videotaped and transcribed. An observer also took detailed field notes (Clark, 1997).

Task 1 (Linear Patterns in Tabular Form Developed in Class)

In class you made a table for the equation $y = 600 - 500x$. you recall the table?

x	y
0	600
1/4	475
1/2	350
3/4	225
1	100

A. What sorts of patterns did you notice in class, and do you notice now? (Do you see a relationship? Is there a pattern here?)

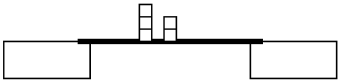
If students notice a recursive pattern, such as y's decreasing by 125:
-- Why does y decrease by 125 each time?
-- Does it matter what the x-values are?
-- Is there a relationship between the 1/4 and the 475?
-- Does your pattern always hold?

B. Do you think this pattern could continue?

C. Can you make additional entries in the table?

Task 2 (Real-World Pattern of Linearity: Bridges and Pennies)

In class you made paper bridges to see how many pennies they would hold. (Say students had developed a particular relationship, such as adding an extra piece of paper to the bridge would result in the bridge holding 10 more pennies).



A. What determines how much weight the bridge can hold?

B. Does it matter how long the bridge is? Would a long bridge hold more pennies, or would a short bridge hold more, or does it not matter?

C. What is the relationship between the layers of paper and the pennies on the bridge?

D. Why does adding an extra layer to the bridge result in holding 10 more pennies? (Could it just as easily be 9 or 11? Will it always have to be exactly 10?)

E. Will this relationship always hold, or does it matter what numbers you use? (Ask about 20 layers, 10,000 layers, and part of a sheet of paper instead of a whole layer).

F. How many pennies would a bridge with 25 layers be able to hold?

G. Say you had 115 pennies that you wanted to put on a bridge. How many layers would your bridge need?

FIGURE 4 Sample interview items.

The primary purpose of a teaching experiment is for researchers to gain direct experience with students' mathematical reasoning, learning, and development (Cobb & Steffe, 1983). The teaching-experiment setting allows researchers to construct models of students' mathematics through the creation and testing of hypotheses in real time while engaging in teaching actions. Through this approach, it was possible to continually develop, test, and refine conjectures about students' generalizations as they solved problems. The teaching experiment occurred on 15 consecutive school days for 1.5 h each day. In order to gain more access into each individual student's understanding, 30-min informal discussions also occurred with 1 student at the end of each lesson, resulting in a total of two discussions with each student.

One goal of the teaching experiment was to explore the nature and development of students' reasoning in the context of realistic problems about linear growth. Three specific aims were (a) to help students develop a ratio as a measure of an emergent quantity such as speed or gear ratios, (b) to provide opportunities for students to create generalizations, and (c) to encourage students to develop appropriate justifications for their strategies, conclusions, solutions, and generalizations. The students were explicitly told that they would be encouraged to explain their thinking and justify their solutions and approaches. Grounded in the hypothesis that meaningful learning must be tied to quantitative referents (Thompson, 1988), the sessions emphasized exploration of the quantities available in two real-world situations involving linearity: gear ratios and constant speed. The students worked with gear ratios for the first 7 days of the teaching experiment, and worked in a speed context for the remaining 8 days. These two contexts were employed in order to encourage the possibility that students would generalize across the two cases in order to develop more global conceptions about the nature of linearity. Because the gears provided a context that previous students had found more tractable in terms of experimenting with different ratios, it occurred first in order to help students develop constant rates of change before tackling the more complex speed situation.

Two physical artifacts ultimately proved important in influencing how the students reasoned. The first was a set of physical gears that the students could directly manipulate in order to experiment with ways of coordinating rotations. The second was a computer program called Simcalc Mathworlds (Roschelle & Kaput, 1996), which simulated speed scenarios showing two characters walking across the screen at constant speeds. The use of the software allowed the students to create and test conjectures about how changing distance and time would affect the character's speed.

Figure 5 provides an overview of the activities students engaged in and the mathematical ideas addressed as they were explored in the teaching experiment.

Day	Mathematical Topics	Class Activities	Context
1	Coordinating quantities	Finding ways to keep track of simultaneous rotations of different-sized gears	Gear Ratios
2	Relating teeth to rotations; Inverse relationships	Determining how to relate the turns of a gear with 8 teeth to a gear with 12 teeth	
3	Constructing ratios; Constant ratios in non-uniform tables	Finding relationships between 8/12/16 gears; determining if rotation pairs come from the same gear pair	
4	Connecting $y = ax$ equations to the gear situation	Explaining how $(3/4)m = b$ relates to both rotations and teeth	
5	$y = ax + b$ gear situations	Modeling situations in which A turns before connecting to B	
6	Representing $y = ax + b$ situations in tables	Making $y = ax + b$ tables; comparing and contrasting to $y = ax$ tables	
7	Non-uniform $y = ax + b$ tables; Isolating quantities for speed	Determining constant ratio from $y = ax + b$ tables; Who walks faster, Clown or Frog	Gear Ratios / Speed
8	Changing initial quantities without changing the emergent quantity	Finding as many ways as possible to make Frog walk the same speed as Clown	Speed
9	Classes of equivalent ratios	Explaining why equivalent ratios mean the same speed	
10	Constant ratios in non-uniform tables	Determining if Frog went the same speed given values in tables	
11	Connecting $y = ax$ equations to the speed situation	Explaining how $(2/3)c = s$ represents speed	
12	$y = ax + b$ speed situations and tables	Modeling situations in which Clown starts away from home and walk constant speed; Making tables to represent $y = ax + b$	
13	Non-uniform $y = ax + b$ tables	Deciding constant speed from $y = ax + b$ tables	
14	Non-uniform $y = ax + b$ tables	Deciding constant speed from tables; describing $y = ax + b$ speed situations	
15	Meaning of Linearity	Inventing situations involving linear relationships	Student Invented

FIGURE 5 Overview of the teaching-experiment unit.

Data Analysis

Analysis of the data from both parts of the study followed the interpretive technique in which the categories of types of generalizations were induced from the data (Glaser & Strauss, 1967; Strauss & Corbin, 1990). The initial coding pass relied on open coding, in which instances of generalization were initially identified as they fit the definition described earlier (the identification of commonality across cases, the extension of reasoning beyond the range in which it originated, and the derivation of broad results from particular cases). Subsequent review of the transcripts of the teaching-experiment sessions, classroom-study sessions, and individual interviews led to the development of emergent categories of types of generalizations. When categories of types of generalizations did not fit with new data, new categories emerged. The categories were then subjected to subsequent passes through both the teaching-experiment data set and the classroom-study data set until theoretical saturation had been achieved. Three complete passes through the data set were required in order for the categories to stabilize, and all generalizations were ultimately re-coded with the final categorization scheme.

Once categories of generalization were identified, videotaped classroom and teaching-experiment data were revisited in order to understand how the instructional environments may have supported and constrained the development of students' specific generalizations. A two-part study was conducted in order to examine the nature of students' generalizing under a more varied set of experiences so as to develop a taxonomy describing as many generalizing acts as possible. The classroom teacher's approach to linear functions included a focus on number-patterns, and one goal of the study was to examine students' generalizations as they reasoned with number patterns. Meanwhile, the teaching experiment provided another way to create, observe, and model the types of generalizations not typically seen in the literature—generalizations related to a focus on quantities. In addition, although the classroom teacher's use of CMP (Lappan et al., 1998) materials suggested the possibility that students might engage in the production of meaningful mathematical relationships, the teaching experiment provided a way to maximize the likelihood that students would focus on quantities and quantitative relationships.

Although the results described in this paper emerged from a two-part study, it was not designed as a comparison study. Conducting a comparison study would have been impractical in part because any comparison between two treatments would require a large number of participants to sustain validity. Working with the number of participants required would have been beyond the scope of a qualitative analysis. In addition, it would have been necessary to develop a study in which the two treatments were as similar as possible in terms of the students, the experience and ability of the teacher, the length and intensity of the lessons, and other factors that would otherwise confound any comparison results. However, ultimately the two-part study did prove serendipitous in allowing for the examination of the relationship between a focus on quantities and the type of generalizations produced. This examination became possible precisely because, in contrast to the teaching-experiment students, the classroom-study students seldom attended to the quantities tied to real-world problems.

Results and Discussion

The results are organized in two sections. In section 1, similarities and differences in students' generalizations across the two-part study are presented and discussed. Section 2 presents a detailed episode in which a student reasoned directly with quantities, but produced generalizations that more closely mirrored those of the students who focused on number patterns alone. The episode is followed by a discussion of the distinction between two types of quantitative reasoning, *direct-measures reasoning* and *emergent-ratio reasoning*.

Section 1: Similarities and Differences in Generalizations

Although students ultimately reasoned with both number patterns and with quantities in both parts of the study, number-pattern reasoning that was disconnected from quantitative referents was prominent in the classroom study and quantitative reasoning was prominent in the teaching experiment. The data revealed four major differences in students' generalizing activity (see Figure 6): (a) the generalizing action of relating occurred more often in the teaching experiment than in the classroom study; (b) the generalizing action of searching occurred in both parts of the study but instantiated itself in different ways, (c) the generalizing action of extending instantiated itself differently in the two parts of the study; and (d) the reflection generalizations of statements of continuing phenomena and sameness occurred more often in the teaching experiment while statements of global rules occurred more often in the classroom study. These differences, coupled with similarities across the two groups, suggest that students' generalizations were closely tied to whether they focused on number patterns or quantities.

A. The Generalizing Action of Relating

When students engaged in the action of relating, they created relationships or made connections between two or more problems, situations, ideas, or mathematical objects. Relating includes recalling a prior situation, inventing a new one, or focusing on similar properties or forms of present mathematical objects. Relating occurred only three times in the classroom study, while it occurred 68 times in the teaching experiment. In an example of a classroom-study case of relating, Carla tried to determine the value of y in a table of ordered pairs given that x was 2/5 (Figure 7). Carla had previously determined that she could find y for $x = 1/4$ by dividing 73 (the direct relationship between x and y) by 4. Carla struggled with $x = 2/5$, in part because she could not appeal to any quantitative understanding of the table. She ultimately decided that this was a similar case to the previous problem in which she had divided 73 by 4. She then converted 2/5 into 0.4, and (incorrectly) divided 73 by 0.4 to obtain 182.5 for y. In so doing

	Classroom Study			Teaching Experiment		
	Int.	Class	Total	Int.	Group	Total
Generalizing Actions:						
Relating Situations, Connecting Back	1	0	1	6	33	39
Relating Situations, Creating New	0	0	0	2	6	8
Relating Objects, Property	0	0	0	0	19	19
Relating Objects, Form	0	2	2	0	2	2
Relating Totals	1	2	3	8	60	68
Searching for Relationships	7	0	7	8	40	48
Searching with Procedures	1	0	1	2	9	11
Searching for Patterns	21	0	21	4	11	15
Searching for Solutions or Results	0	1	1	0	2	2
Searching Totals	29	1	30	14	62	76
Extending by Expanding Range	0	0	0	1	2	3
Extending by Removing Particulars	0	0	0	1	14	15
Extending by Operating	8	1	9	7	8	15
Extending by Continuing	22	0	22	3	20	23
Extending Totals	30	1	31	12	44	56
Reflection Generalizations:						
Continuing Phenomena	1	2	3	13	37	50
Sameness, Common Property	6	1	7	21	53	74
Sameness, Object or Representation	2	0	2	0	22	22
Sameness, Situation	0	1	1	5	18	23
Sameness Totals	8	2	10	26	93	119
General Principle, Rule	13	12	25	18	77	95
General Principle, Pattern	35	13	48	13	50	63
General Principle, Strategy or Procedure	0	1	1	6	13	19
General Principle, Global Rule	39	2	41	14	12	26
General Principle Totals	87	28	115	51	152	203
Definition, Class of Objects	0	1	1	2	4	6
Influence, Prior Idea	3	0	3	6	30	36
Influence, Modified Idea	2	0	2	0	5	5
Influence Totals	5	0	5	6	35	41
Totals	161	37	198	132	487	619

FIGURE 6 Counts of students' generalizations in the teaching experiment and the classroom study.

Carla established a relationship between the two problems, and then implemented her numeric strategy to the new problem in order to determine y for $x = 2/5$.

When the teaching-experiment students engaged in the generalizing action of relating, they established relationships of similarity between two situations, such as the speed situation and the gears situation, or between one of those situations and a prior personal experience. The prominence of relating actions was likely tied to both the students' real-life familiarity with experiential qualities as they

x	y
0	0
1	73
2	146
3	219
4	292
5	365

FIGURE 7 Carla's table of x- and y-values.

reasoned with quantities in mathematical situations, and to the manner in which the teaching experiment was run. For example, when working with gears, the students tried to explain why a large gear made fewer rotations given the same turning speed as a small gear. Several students related the rotations of the gears, which they could visually observe, to prior experiences of running in circles around a track:

> Timothy: [The big gear] would turn less because it has a bigger circumference to make a whole turn. And the small one has a smaller circumference to go around.
>
> Julie: 'Cause like if you were to take the circumference of the two, and lay them out, this one would be shorter, be like a shorter distance to run or something. 'Cause like, when people are running a track, um, it's luckier if you're um, on the inside of the track because, it doesn't—it's not a, such a big lap, than being on the outside.

The students could relate their understanding of the rotations of a gear to their experience of running around a track at a given pace, and more generally, the students' actions of focusing their attention on relationships between quantities appeared to trigger memories of previous experiences with similar qualities.

The design of the teaching experiment deliberately facilitated acts of relating by immersing students in two quantitatively-rich situations, gears and speed. The speed scenario immediately followed the gears scenario, and students encountered a number of problems with similar structures or numbers in both contexts. These similarities encouraged the creation of connections between gears and speed. In particular, when the students moved to speed situations, they remained focused on the invariant ratio between centimeters and seconds, which prompted the recall of another invariant ratio—the rotations of one gear to the rotations of another gear. Because both gear ratio and speed are instantiations of a constant multiplicative relationship, students may have been able to relate their current activity to prior activities, detecting a similarity between the two situations.

In contrast, when students in both parts of the study focused on number patterns, they did not have a way to connect their current number-pattern reasoning to prior experiential familiarity with quantities. In particular, the classroom-study teacher seldom prompted the students to construct ratios or reflect on constant multiplicative comparisons. Thus it may have been more difficult for the students to notice similarities between tables when the only similarity was an undetected constant ratio (or ratio of differences) between x and y. The students did not focus on these. Instead, they focused on the different types of patterns they found in the tables. A pattern such as "when x goes up by 5, y goes up by 8" might not have seemed similar, from the students' perspective, to a pattern such as "y increases by 10 each time."

The teaching-experiment students' frequency of relating both situations and objects led to many reflection generalizations on sameness and the influence of prior ideas, which is not seen in the classroom-study students (see Figure 6). By relating either situations or objects, they focused on what was the same across those situations and objects, which often led to the use of a previously developed strategy. The teacher-researcher also explicitly encouraged those connections, often by asking students to reflect on what was the same or different across problems or even mathematical objects within a problem. For instance, when one student graphed points relating various distance-time pairs for a character walking a constant speed, the teacher-researcher asked, "And what about two different points on the line, like say, this point and this point. What is the same about these points?" Timothy answered, "y is always 4/5 of x. It's the same." This is an identification of a common property across the points, and the discussion then veered towards questioning whether it would be appropriate to extend the graph into negative x- and y-values. As the students debated whether one could have negative distances and times, they then began to engage in relating actions by recalling other situations with negative values.

B. The Generalizing Action of Searching

Students engaged in the generalizing action of searching for sameness in both parts of the study, although in the classroom study searching was observed almost exclusively in individual interviews. The major difference occurred in what types of searches the students conducted. Those who focused on number patterns alone, mainly but not exclusively in the classroom study, focused their searches on patterns, while those who attended to quantities, mainly in the teaching experiment, focused their searches on relationships.

The searching counts in Figure 6 can be somewhat misleading. Although the number of videotaped hours was roughly equivalent for both parts of the study, the data from the teaching experiment were richer and more abundant. The classroom-study students did not have many opportunities to publicly share their generalizations in class, and therefore the data on their generalizing was largely limited to one-on-one interviews. As a result, the differences in numbers can be attributed in part to the fact that there were three times as many generalizations detected in the teaching-experiment data. Nevertheless, the scarcity of searching for the same relationship in the classroom study suggests a substantive difference despite the overall difference in total cases of generalizing, and the greater frequency of searching for the same pattern in the classroom study is also noticeable given that the total number of generalizations overall was so much smaller.

Searching for patterns. When searching, students performed a repeated action in order to determine an element of similarity. Searching for the same pattern involved identifying a pattern and then seeking to determine whether it remained stable. For instance, the classroom-study student Carla examined the following table (Figure 8):

x	y
0	0
1	2
2	5
3	9
4	14
5	20

FIGURE 8 Table of non-linear x- and y-values.

Carla: I kind of see a pattern between the *y*, but not the two numbers, like *x* and *y*.
Int: What do you see between the *y*?
Carla: Um, I see that it jumps by 1 each time you go up. Because like it's, here wait. Because 9, because 14 minus 9 is 5, but then 20 minus 14 is 6.
Int: Uh huh.
Carla: And so I think that it jumps 1 in the *y* category. So that it's like the last number plus 1.

Carla could not identify a relationship between *x* and *y*, but did locate and generalize a stable pattern in the table, which was not connected to any quantities. Because the participating teacher had introduced different types of number tables in the classroom on the day Carla was interviewed, the interviewer used some of those tables as well as other quantitative situations to examine Carla's thoughts on linearity. The teacher's emphasis on well-ordered, naked number tables was oriented towards helping students understand the need to attend to the recursive rates of change for *both* *x*- and *y*-values, since previously the students had erroneously focused only on the *y*-values. The teacher's provision of many practice situations in working with a variety of different tables helped the students develop a fair amount of proficiency with noticing and searching for patterns, but these patterns did not necessarily represent quantitative relationships.

Searching for relationships. The teaching-experiment students worked with the same table of ordered pairs (Figure 9) in two different contexts. In the first context the values represented gear rotations and the students initially focused only on the number patterns in the table. At this stage, the teacher-researcher did not heavily intervene but instead allowed the students to work together to make sense of the table in small groups. The decision to introduce the same table within the speed context was prompted by a desire to help the students make a connection between the two situations, gears and speed, and thus focus on the quantities themselves rather than the numbers alone. Therefore, a week later, the students encountered the same numbers once again, but this time the left column represented

Small	Big
7 1/2	5
27	18
4 1/2	3
16	10 2/3
1/10	1/15

FIGURE 9 Table of gear rotations.

the number of centimeters a character walked, and the right column represented the number of seconds it took to walk those centimeters. The teacher-researcher decided to ask the students to determine if the character appeared to walk the same speed throughout the table, or if he sped up or slowed down. The nature of this question further supported an emphasis on quantities rather than numbers alone. As a result, the students approached the table in a way that characterized a focus on relationships between the quantities rather than a focus on number patterns:

> Timothy: I found that no matter what, the seconds over the centimeters is 2/3. So he's walking the same pace.

Timothy took the seconds in each pair and divided by the centimeters, and found that each time the result was two-thirds. The teacher-researcher pushed him to describe how he had concluded that the character walked the same speed, and Timothy explained:

> Timothy: Because Clown is walking 2/3 of a second for every centimeter he walks. And so every single time it's the same thing.
> Teacher: Now, I was asking you to look at this table to see if all of the pairs were the same. What is sameness here? What does same mean?
> Larissa: The relationship.
> Timothy: They all have the same relationship between seconds and centimeters.
> Teacher: Which translates to what?
> Timothy: That either if you look at it centimeters to seconds, the centimeters are 1 and 1/2 of the seconds, or the seconds are 2/3 of the centimeters.
> Teacher: Okay. And what does sameness mean in terms of the situation?
> Larissa: It's, he's going the same speed throughout the whole time.

Timothy's search for sameness focused on the relationship between centimeters and seconds, and this relationship appeared to be tied to his quantitative comprehension of the situation. By identifying the invariant ratio across the different pairs, Timothy was able to reflect on the different number pairs representing the same emergent quantity, which supported his eventual development of an equivalence class of ratios. This process exemplifies the more general trend in which students who focused on the quantities represented by numbers generalized by searching for and finding relationships between those quantities. In addition, the teacher-researcher's emphasis on pushing the students to describe what sameness meant in terms of the situation further encouraged students to focus on relationships rather than just patterns.

C. The Generalizing Action of Extending

Extending by expanding and removing particulars. While students from both parts of the study engaged in the action of extending, there were no instances of extending by expanding or by removing particulars in the classroom study. However, the fact that these actions manifested themselves only in the teaching experiment is probably more strongly related to the teacher-researcher's actions than to whether the students were reasoning with quantitative relationships. When students extend by expanding the range of applicability or by removing particulars, they note that a property, pattern, or relationship can be applied to a larger range of cases. This action might occur through extending one's reasoning within the same problem or into a new area. It could also include the deliberate removal of contextual details from a description of a phenomenon in order to state a global case.

None of the students in either group appeared to be naturally inclined to extend their reasoning in these manners, regardless of their focus on quantitative relationships. When the teaching-experiment students did extend by expanding or removing particulars, they generally did so at the teacher-researcher's urging. The teaching-experiment students faced a continued social need to generalize their reasoning; the classroom culture was centered on promoting and valuing generalizing and justifying actions. Thus the teacher-researcher would sometimes ask the students to generalize an idea, relationship, or strategy to any two gear pairs, or to any characters walking the same speed. Through deliberate concentration, the students could do this, but these actions did not occur spontaneously. As an example, let us return to the episode described in the Relating section in which a student graphed points relating various distance-time pairs for a character walking a constant speed. After the student identified the slope of various graphs as representing speed, the teacher-researcher asked the student to generalize his argument:

> Teacher: Timothy, can you make a general statement about how the slope of a graph, any graph, is related to the speed of a character, any character?
> Timothy: Um, they're basically going to, as long as the graph is linear, uh, as long as the uh rule is linear, um, it's always gonna be whatever of x is gonna equal y, or whatever of y is gonna equal x.
> Teacher: And what's that got to do with speed?
> Timothy: Yeah. Um, basically it's gonna be continuously the same fraction for the slope. And basically that means it's gonna be the same units per whatever, uh per whatever amount of time.

Extending by continuing and operating. Students who located number patterns, in both parts of the study, could often extend those patterns to generate new cases, either by continuing them or by operating on the patterns (for instance, by doubling or halving a given ratio to create new ordered pairs in a table). Brianna, a classroom-study student, demonstrated both generalizing actions as she worked with the following table (Figure 10):

After having searched for a pattern and identified that y increases by 7 for each unit increase in x, Brianna continued the pattern to generate several new

X	y
0	5
1	12
2	19
3	26
4	33
5	40
6	47

FIGURE 10 Brianna's table of linear values.

pairs: "If the next one was 7, the next one would be . . . 54. If it was 8, it would be . . . 61."

Brianna then operated on the 1:7 increase in order to generate the pair ($1\frac{1}{2}$, $15\frac{1}{2}$):

> Brianna: If, on these you're adding 7. I just . . . added . . . 7. Oh. I added 7 but I took the halves, so it'd be 1 and a half, and half would be, like 1. So um, I took the halves and if you added 7, that'd be 12. I just took the halves, and 7, so it'd be . . . uh, 7/2 is . . . 3 and a half. Yeah. So I just added 3 and a half to 12.

Because the generalizing actions most prominent when students focused on number patterns were the search and location of patterns and procedures, it follows that students also focused on those very patterns when extending their reasoning.

D. Reflection Generalizations: Statements of Continuing Phenomena, Global Rules

Continuing Phenomena. Only three statements of continuing phenomena were recorded in the classroom study, while 50 such statements appeared in the teaching experiment data. The prominence of the statements of continuing phenomena in the teaching experiment may be tied to the ways in which the students reasoned with quantities. A statement of a continuing phenomenon reflects a focus on the dynamic relationship between quantities; it is characterized by a sense of continuation, motion, or extension. For instance, when working with the table in Figure 9 in the speed setting, Dora remarked "for each centimeter it takes the clown 2/3 of a second." Similarly, when describing a pattern in gear rotations, Timothy explained "The big one moves 2/3 of a turn every time the small one

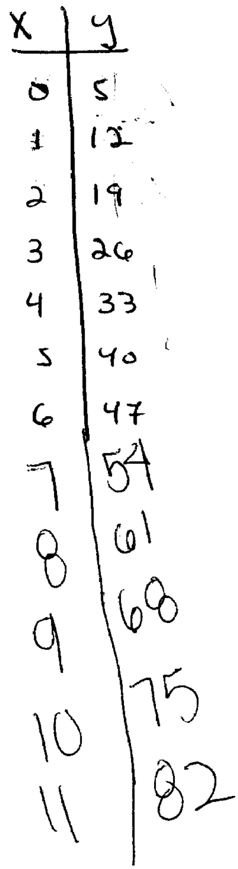


FIGURE 11 Brianna's extended table.

turns a full turn." These continuing phenomena statements indicate that students could form an image of two co-varying quantities (Thompson, 1994).

The teaching-experiment students reasoned with the emergent quantities speed and gear ratios. Both of these quantities reflect a dynamic relationship between two initial quantities. Speed is characterized not only by a ratio relationship between distance and time, but it is also a continuous phenomenon. The students had prior experience with speed as continuing motion through space, and in class they watched characters walk across a computer screen at different speeds. When the students in the teaching experiment turned their attention to comparing centimeters to seconds in order to reason about speed, they referenced the dynamic nature of speed in their generalization statements. Similarly, the gear-ratio quantity was one that students could observe by turning attached gears of different sizes. They could see this quantity by observing how much faster one gear would turn when it was attached to a larger rather than a smaller gear. When the students turned the gears, they did so by spinning a handle smoothly around; they did not stop the rotation after each gear had made a complete turn. Thus the gear ratio was characterized by smooth, continuous motion. This dynamic understanding made its way into the students' language as they produced statements of generalization.

The prominence of statements of continuing phenomena in the teaching experiment could have been connected to the type of quantities with which the students reasoned as well. For instance, the quantity of steepness, as related to the ratio of height and length, does not suggest the same type of continuous motion as suggested by speed or gear ratios. Similarly, when the students in both parts of the study reasoned with number patterns alone, they most frequently did so in the form of tables of numbers. These tables contained patterns that did not suggest a sense of continuation; instead the students detected and stated static patterns.

Statements of general principles: global rules. The classroom study students demonstrated a high incidence of global rule generalizations in their individual interviews. This frequency could have been related both to their focus on numbers and to the participating teacher's instructional style. The students' focus on numbers resulted in the construction of generalizations that were tied to the nature of the number tables they encountered. For instance, Juliana stated that a table of ordered pairs only represents linear data if "it's in a continuous pattern that's the same every time." She had only encountered well-ordered tables in class, and when she and her classmates collected real-world data such as the number of layers of paper in a bridge and the number of tiles it could support, they compiled their data in well-ordered tables and then almost immediately moved towards focusing on patterns in the table, rather than on the situation represented by the table. Lacking opportunities to explore how quantities changed in relationship to one another, the students developed global rules that depended on particular organizations of data.

Moreover, the participating teacher encouraged a focus on patterns and the creation of general rules about those patterns—in fact, the only significant instances of classroom study student generalizing recorded in the classroom, in contrast to during the individual interviews, occurred in the "general principles: pattern" and "general principles: rule" categories. For instance, the participating teacher (PT) introduced a well-ordered table of data comparing the cost of an item and its post-tax price, and asked students to find patterns:

PT: Now. Let me ask you this. Did anybody see patterns? See if your group can articulate a pattern.
Miguel: Yeah, it went up by 1.08.

PT: All right, did this group find a pattern? Mario, let's hear about your pattern.
Mario: You go up 1.08 each time.
PT: You did what?
Mario: Add.
PT: Add 1.08. 1.08 + 1.08 + 1.08 + 1.08. So there's a pattern in the y-column, yes. Who has something slightly different? All right, Bob.
Bob: You can multiply x by y, no, x times 1.08.
PT: You can multiply x times 1.08. All right. That's certainly something we noticed.

The participating teacher ultimately encouraged students to make decisions about whether data were linear based on more general rules: "Now let me ask you this question. Each time, is the price rising by a steady amount? Yes, everybody's sure of that. Stand up if you think this situation would graph as a line." All of the students stood, and the teacher ended by explaining that a steady increase indicates linear data. The classroom emphasis on finding patterns and then developing rules about linearity was reflected in the students' generalizing in individual interviews.

Statements of general principles: patterns. One similarity that emerged in students' reasoning across both parts of the study was that they produced statements of general patterns when they attended to number patterns. In many cases students stated the patterns they had identified, often as a result of the generalizing action of searching for the same pattern. For instance, when Brianna first encountered the table above (Figure 10), she searched for a pattern and then remarked "You're always adding 7 on the y side and you're always adding 1 on the x side." Similarly, when Carla stated "it's like the last number plus 1," her statement was the reflection generalization of identifying the general pattern, which mirrored her generalizing action of searching for the pattern.

The differences that emerged across the two-part study did not exclusively fall along the boundaries of "classroom-study generalizations" versus "teaching-experiment generalizations." Instead, when students across both parts of the study reasoned with number patterns, they searched for patterns, extended those patterns by continuing them or operating on them, and they developed final statements of general patterns. When students focused on quantities, they were more likely to engage in relating actions, they searched for relationships, and they developed statements of continuing phenomena. However, there was one instance in which a classroom-study student attended to quantities and quantitative relationships, but his generalizations were more similar to his classmates who attended only to number patterns. This instance is described in Section 2, and is followed by a discussion of the importance emergent-ratio quantitative reasoning in influencing students' generalizations.

Section 2: Direct-Measures Versus Emergent-Ratio Quantitative Reasoning

Although both groups of students occasionally focused on quantities and their relationships, the *type* of quantitative reasoning in which students engaged ultimately proved more important in influencing their generalizing than a simple distinction between a focus on quantities versus number patterns. The following episode presents a case in which a classroom-study student engaged in a type of quantitative reasoning that resulted in generalizations very similar to those seen by students who focused on number patterns alone. It is included to highlight the difference between two aspects of quantitative reasoning, direct-measures reasoning and emergent-ratio reasoning, and to demonstrate the importance of these aspects in influencing how students generalize. In the episode, Ricardo reasons with two quantities in a simulation of a real-world situation—but he does not construct a third emergent quantity as a ratio of the two initial quantities. The episode demonstrates Ricardo's difficulty making logical inferences about the relationships in the problem. As a result, his generalizations are incorrect statements of general rules and patterns.

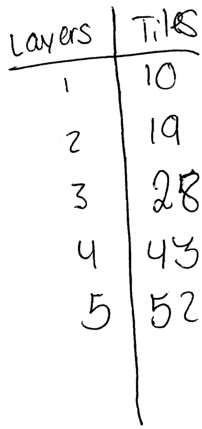
The Episode: Bridges and Tiles

The classroom study students worked with a CMP-derived activity called "strength of bridges." Students built bridges from sheets of notebook paper, and compared the number of layers of paper in the bridge with the number of tiles the bridge could hold. Ricardo's group had found that when comparing the number of tiles a bridge could hold with how many layers the bridge consisted of, there was a roughly stable relationship between the layers and the tiles. They concluded that each additional layer could hold seven additional tiles, which they formalized as $y = 7x$, where x is the number of layers and y is the number of tiles.

In his individual interview, Ricardo stated a general pattern: "The more layers you put, the more tiles you could probably hold." Reasoning with the quantities of the number of layers and the number of tiles, Ricardo created a table that showed his opinion of what the values might look like if his group had been able to take perfect data without any measurement error (Figure 12):

When asked if he saw a pattern or relationship in the table, Ricardo responded, "Well, I would think each one, like, each one . . . hmm. I'd say about near 10 or 11 or 12 tiles are, each time probably are added. 10." Ricardo's new reflection generalization was another more specific statement of a general pattern. He wrote out a new table showing what he meant (Figure 13):

Ricardo explained why he made a change in his table at 4 layers:

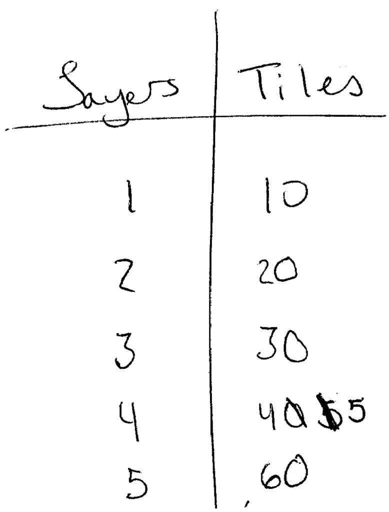


Layers	Tiles
1	10
2	19
3	28
4	43
5	52

FIGURE 12 Ricardo's initial table of layers and tiles.

Ricardo:But I was thinking at five it, since there were so many layers, it'd probably be more than double. Like, probably, four and five. So I would say this would equal about, probably 15 up. And this, probably well 60 if it was evenly.

By "more than double", Ricardo referred to the idea that the tiles would increase by more than 10 for an additional layer. To the reader, a pattern of adding



Layers	Tiles
1	10
2	20
3	30
4	40 45
5	60

FIGURE 13 Ricardo's revised table of layers and tiles.

10 each time is different from a pattern of doubling each time. However, for Ricardo, “adding 10” and “doubling” appeared to be identical. So he again revised his statement of a general pattern to one in which the tiles eventually begin to increase by more than 10 for each additional layer. Ricardo explained, “The paper, see, one, two and three, there’s really not much of a difference between the layers. But as you go on to four, it gets more thick, and then five, even thicker.” While Ricardo began to form a more precise relationship between the two quantities, this relationship was not a ratio. Note, however, that Ricardo repeatedly referenced the idea of number of layers and thickness, indicating that his mathematical focus was on those quantities rather than on the numbers alone as seen in the tables he created. Moreover, he reasoned with mental images of the way those quantities would change, and generalized based on his images.

The following question was posed to determine the robustness of Ricardo’s pattern:

> Int: So what if you had a whole bunch of layers, like what if you had 50 layers?
> Ricardo: Well it’s gonna be kind of hard to hold on this type of paper. But if it were a lot bigger, it’d probably be able to hold, hmmm. Probably close to 690 if it were really big. For 50 layers.
> Int: How’d you come up with that?
> Ricardo: Well, 5 times 10 is 50, so 600. And I added a little bit more because you add more as you go up.

The last pair in Ricardo’s table was (5, 60). After multiplying 60 by 10 to get 600, Ricardo appeared to add the extra 90 tiles because he did not think the relationship should be linear. He also shifted from thinking about adding 10 tiles for each layer to multiplying by 10 before compensating with the extra 90 tiles. His reasoning has shifted from an additive to a multiplicative comparison, but it remained a within-measures comparison rather than a between-measures comparison (Vergnaud, 1983).

> Int: So would that continue if you had . . . say you had something huge, like 5,000 layers?
> Ricardo: Hmm. It’d probably go up more than 10, I think.
> Int: Since it’s so many layers?
> Ricardo: Uh huh.
> Int: How much, do you think?
> Ricardo: I’d say about probably 30, in one stack. It could probably go up, probably even 50 or 60.

By “go up”, Ricardo appeared to mean two things: (a) increasing by that many from one layer to the next, and (b) multiplying the number of layers by the value such as "50 or 60." When asked to come up with a number, Ricardo wrote "7100 tiles." Ricardo explained his reasoning as follows: "I mean 50, I multiplied it by 100 to get 5000. So I multiplied 690 by 100 and added 200 to it." Ricardo multiplied incorrectly by a factor of 10, instead multiplying 690 by 10 to get 6900. The addition of 200 extra tiles was due to Ricardo's belief that the relationship should not be constant. Note that as Ricardo focused on extrapolation, his attention shifted from thinking about layers and tiles to the numbers themselves. His language now appears more similar to those of the students discussed above who focused on number patterns, such as Carla.

When asked to state a direct relationship between the number of layers and the number of tiles, Ricardo explained:

> Ricardo: Probably from one, two and three you could do, um, 1 times 10 equals, like, you can multiply by 10, 'cause it goes up by 10.
> Int: What would that look like, just for the 1, 2 and 3, if you wrote that?
> Ricardo: You could just put 1 times 10 equals 10. Two times 10 equals 20. If you go on, it's, hmm, equals . . . I'm just gonna put it like that. [Writes "$4 \times 10 + x = y$."]
> Int: Okay. So you wrote 4 times 10 plus x equals y. So what does that mean?
> Ricardo: Because I'm not really sure what to put here (points to the x) because it's gradually going up. So I just put that and it can equal anything if the x equals something different than y, can equal . . .

Ricardo first stated the reflection generalization of a statement of a general pattern, that "it goes up by 10." He also attempted to describe the "adding a few extra" idea he had incorporated throughout the episode with x, culminating in the reflection generalization of the general rule $4 \times 10 + x = y$. He was then asked what x should be for 4 layers:

> Ricardo: I just added an extra 5 in there 'cause, I didn't add any 5's here, I just added 10, um, times, to that by 10, so, it should equal 45.
> Int: Ah, I see. And so, could you use this equation that you wrote, 4 times 10 plus x equals y, for the 5th layer?
> Ricardo: Uh huh.
> Int: How would that work?
> Ricardo: It'd be 5 times 10 plus x equals y, but then the x would equal 10. I think it probably gradually, you add. Okay it goes like, you add 5, then 10, then 15, then 20. Probably like that.

Ricardo's new general rule was that x could increase by five each time, which represents a further formalization of his earlier attempts to compensate. Further, he was able engage in the generalizing action of extending that idea by continuing the pattern to create new pairs. While Ricardo's rule for 4 or more layers is actually linear ($T = 15L - 15$, for $L \geq 4$), there is no evidence that he saw it as such. After using his equation to determine how many tiles 50 layers would hold, Ricardo was asked to justify his equation, which prompted a return to thinking about quantities rather than just numbers:

> Ricardo: Well, to me I think it is, in my opinion. But other people might think different.
> Int: Okay, and how could you justify your opinion?
> Ricardo: Well, I could probably try and do the um, try to make another bridge, and also do that over. But it wouldn't work for 5000 or 50 because you probably couldn't put that many tiles. You wouldn't have that many tiles.
> Int: Is there any other way you could justify besides actually making the bridges and testing them?
> Ricardo: Well, there's probably an equation to probably be solved but I'm not really sure what it is right now.

Reasoning with Two Quantities versus Three: The Importance of Emergent-Ratio Quantitative Reasoning

The bridges and tiles episode demonstrates the importance of the *type* of quantitative reasoning in which a student engages. Ricardo reasoned with two quantities, layers and tiles. He was able to articulate that there was a relationship between layers and tiles, and could understand that more layers would result in more tiles. Ricardo also made attempts to quantify this relationship, and he did eventually make within-measure multiplicative comparisons between layers and layers, and between tiles and tiles. In contrast, the teaching-experiment students reasoned with three quantities: two initial quantities, such as distance and time, and a third, emergent quantity that the students constructed as a ratio. Once the students had constructed the ratio, they could reason *directly* with the third quantity, such as speed.

Ricardo, however, did not construct a ratio to reason directly with the emergent quantity comparing layers and tiles. The quantity that would compare the ratio of the thickness of a bridge (as measured by the number of layers) and the weight it can hold (as measured by the number of tiles) represents the load-carrying capacity, or endurance, of the bridge. Reasoning directly with a third quantity such as endurance constitutes a different aspect of quantitative reasoning, which can be termed "emergent-ratio quantitative reasoning". In contrast, Ricardo engaged in "direct-measures quantitative reasoning" by focusing only on the two quantities that could be measured directly, layers and tiles. While direct measures quantitative reasoning may involve multiplicative comparisons, these comparisons do not lead to the construction of a ratio as the emergent quantity.

Although Ricardo reasoned with quantities, his generalizing took on the characteristics of those who reasoned with number patterns alone: he engaged in the generalizing actions of searching for patterns, extending by continuing patterns, and he produced reflection generalizations that were statements of general patterns and rules. What could account for this similarity? When students focused on number patterns, they reasoned with two conceptual objects, namely two types of numbers, such as x-values and corresponding y-values. Similarly, Ricardo reasoned directly with two quantities, the number of layers and the number of tiles. Unlike those who engaged in emergent-ratio quantitative reasoning, Ricardo did not construct a third quantity as a ratio of layers to tiles. This difference could explain why the generalizations Ricardo produced mirrored those seen when students reasoned with number patterns more than those produced by students who reasoned with emergent quantities. The distinction affecting students' generalizing can therefore be described not in terms of quantitative reasoning versus number-pattern reasoning, but instead in terms of whether students reasoned with two quantities (or numbers), or whether students constructed and reasoned with a third, emergent-ratio quantity.

In order to develop a quantitative operation, one must conceive a new quantity in relation to one or more already-conceived quantities. Students reasoning with speed can think about measuring "fastness" as a relationship between distance and time. The teaching-experiment students who reasoned with the emergent-ratio quantity of gear ratio, which they described as "the relationship," could think about the dependency relationship between the rotations of one gear and the rotations of another gear. The choice of these two contexts was deliberate because in both cases, students could experiment with how changing either of the two currently existing quantities would affect the new quantity. For instance, problems were specifically designed to require students to make the frog walk the same speed as the clown, or walk twice as fast or one-fourth as fast as the clown, or determine if the frog walked the same speed or at varying speeds given many different centimeter-second pairs. These activities both necessitated the creation of a ratio as a measure of the emergent quantity, and allowed the students to experience that ratio remaining constant even as the initial quantities changed.

For example, in one episode students were asked to find different ways to make Frog walk the same speed as Clown, when Clown walked 15 cm in 12 s (for a detailed description, see Ellis, 2007b). One group of students guessed 10 cm in 7 s, and by watching a SimCalc movie of the characters walking; they found that while Frog and Clown appeared to walk close to the same speed, their speeds were not identical. They then tried 10 cm in 8 s, which did result in the same speed. Eventually, the students worked with multiples of 5 cm in 4 s, finding that pairs such as 30 cm in 24 s and 10 cm in 8 s achieved their goal. The students were then able to create arguments for why 15 cm in 12 s was the same speed as 5 cm in 4 s. In one instance, Larissa and Maria drew a proportional drawing comparing Frog walking 5 cm in 4 s, and Clown walking 15 cm in 12 s (see Figure 14). Larissa explained:

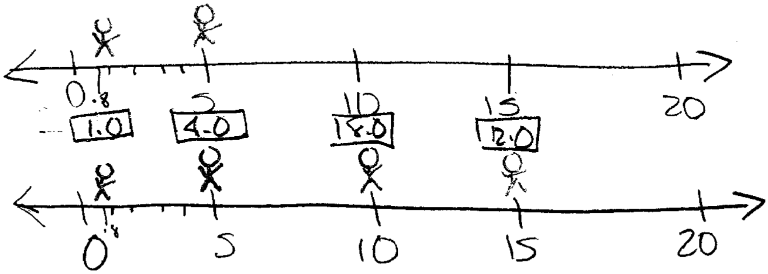


FIGURE 14 Larissa and Maria's proportional drawing comparing Clown and Frog's journeys.

> Larissa: Because for the, when they're at 4, both of them are at 4 s. But since the frog stops, he's finished. So he's finished at 4 s. But the clown keeps going and from 0 to 5 it jumped 4 s, from 4 to 10, and from 5 to 10 it also jumped 5 cm and 4 s. And from 10 to 15, it jumped 5 cm and also 4, um, s. So the proportion stays the same throughout the thing even though Frog stopped.

Larissa's explanation reveals that she was able to form a composed unit of 5cm:4s, and then iterate that unit to make 15 cm in 12 s. (Her use of the word "jump" did not refer to the frog's motion, which was smooth, but instead referred to the increase in the numbers between each successive box in the figure). This composed unit represented speed for Larissa and others, and the students then acted on that unit in ways that helped them generalize speed as a ratio of distance to time.

Ricardo's inability to conceive of the linear relationship as an emergent-ratio quantity of endurance was likely tied to the lack of these types of opportunities. During the bridges activity, the participating teacher articulated a goal that students generalize the linear relationship that the bridge could hold seven additional tiles for each additional layer. There is little evidence, however, that what was a ratio for the teacher, 7 tiles:1 layer, was necessarily a ratio for Ricardo. Ricardo lacked opportunities to observe or experiment with the quantities and their relationships in a way that either required the construction of this ratio, or supported conclusions that the ratio must remain constant regardless of the total number of layers and weights. Therefore Ricardo did not conceive of endurance as an emergent quantity in constant ratio to thickness and weight, and he did not appear to have any way to build conceptions about how endurance could be determined by relating thickness and weight. In addition, because the bridge's endurance was not a quantity that the students could visually perceive (unlike speed or a gear ratio), Ricardo may not have been able to isolate endurance as a ratio as the important quantity. In this case, it is unlikely that he would have then been able to generalize based on a quantity he had not isolated.

As the episode progressed, Ricardo may have shifted his focus from the quantities to the numbers because he lacked the emergent ratio of endurance to reason with directly. Without the language or experience to identify or directly conceptualize the notion of endurance, Ricardo was forced to continue reasoning with two conceptual objects, the number of layers and the number of tiles. This condition meant that his generalizing took on qualities similar to other students who reasoned with two conceptual objects rather than three, namely, the students focused on number patterns.

The effect of reasoning with emergent-ratio quantities can shed light on why students' generalizations differed according to their focus on number patterns versus quantitative relationships. The prominence of relating actions in the teaching experiment was likely tied to the students' real-life familiarity with experiential qualities as they reasoned with quantities in mathematical situations. Because both emergent quantities, gear ratio and speed, are instantiations of a constant multiplicative relationship, students were able detect similarities between the two situations. In contrast, when students attended to number patterns, the patterns did not reference the real-world quantities from which the problem might have been drawn. Therefore, the students did not have a way to connect their current number-pattern reasoning to prior experiential familiarity with qualities or quantities.

When searching, recall that the teaching-experiment students more often searched for relationships, while the classroom-study students more often searched for patterns. When students directed their attention to the emergent quantities of speed and gear ratios, these quantities could only be represented in relationship to the existing quantities instantiated in a table. Thus it was natural to seek out those relationships. If the numbers did not represent quantities, however, there was no motivation to multiplicatively compare them in order to determine a relationship between them. The students did not perceive any reason to consider a pre-existing relationship, such as speed, to investigate. Instead, it was more natural for them to seek out regular patterns in the table, most often by looking at successive differences. Thus students searched for, found, extended, and made general statements about patterns rather than relationships.

The prominence of continuing phenomena connected to reasoning with quantities could also be explained by the construction of an emergent ratio. A continuing phenomenon typically reflects an appreciation of the co-varying nature of quantities, which may be clarified for students once they have constructed ratios and observed those ratios remaining constant. In absence of these experiences, the classroom-study students relayed statements of global rules that they developed from their pattern-seeking activities.

Concluding Remarks and Implications for Instruction

A focus on emergent-ratio quantities was connected with different types of generalizations than those tied to number-pattern reasoning, or reasoning related to only two quantities rather than three. When students focused on emergent-ratio quantities, their generalizing actions included relating and searching for the same relationship, rather than searching for patterns and procedures. The reflection generalizations students produced were statements of continuing phenomena rather than statements of global rules. Moreover, the generalizations connected to reasoning with emergent-ratio quantities proved more productive for students in terms of developing correct extensions, creating correct global conclusions about the nature of linearity, and producing appropriate justifications. These results provide further evidence to hypothesized claims that if mental actions are tied to experience, any meaningful learning must be originally grounded in quantitative referents (Hirsch, 1997; Kaput, 1999; Steffe & Izsák, 2002; Thompson, 1994b, April).

More importantly, the study's results emphasize the need to move beyond a simple quantitative reasoning/number-pattern reasoning distinction when considering how students develop ideas in algebra. Previous work has identified both instances in which focusing on quantitative relationships encouraged the development of meaningful concepts and generalizations (e.g., Curcio et al., 1997; Ellis, 2007b; Hall & Rubin, 1998; Lobato & Siebert, 2002), and instances in which placing students in quantitatively-rich situations did not guarantee that they would create algebraically-useful generalizations (Nobel et al., 2001; van Reeuwijk & Wijers, 1997). This study suggests the need to more carefully consider the *type* of quantitative reasoning in which students engage. Although reasoning with quantitative relationships can support more sophisticated mathematical activity, students who fail to create new mathematical objects, such as emergent ratios, may not gain any additional benefit from focusing on real-world quantities. This result suggests the need for future work identifying relationships between the ways in which students reason with quantities and the concepts and generalizations they develop.

These results also suggest some implications for instruction. First, they highlight the importance of the type of situation used to explore ideas of linear growth as well as the ways in which data are presented. Specifically, teachers should consider the value in developing linearity as an experiential, emergent quantity. Approaches that address linearity through number patterns embody the expectation that students will form a numeric ratio and notice the constant nature of the ratio. However, in pattern situations in which the two sets of numbers do not represent quantities, students cannot easily conceive of the numeric ratio as its own phenomenon. They will have nothing to compare it to, and may lack opportunities build on their experiences in order to construct meaning for constant ratio. In absence of this meaning, students risk generalizing based on unproductive, coincidental patterns that are artifacts of particular organizations of data, as has been reported in the literature.

Students should therefore be confronted with problem situations that require them to explore the phenomenon in question; they should have opportunities to engage in activities such as (a) exploring how changing one or both initial quantities will affect the emergent quantity, (b) determining how to adjust the initial quantities while keeping the emergent quantity constant, and (c) determining how to double, halve, or otherwise manipulate the emergent quantity in relationship to the initial quantities. Of course, helping students focus on quantitative relationships of these kinds is not easy or automatic; these types of activities require a classroom shift that may take significant time and effort, given the current school mathematics emphasis on numbers and patterns.

Given the importance of experiential, emergent quantities for the initial construction of ratio, curriculum designers and instructors should focus on situations that support these conceptions and avoid ones that do not. Some problem situations pose contrived contexts in which data are not naturally linear, presenting relationships such as how many surfboards are sold for a given temperature at the beach. The contrived nature of these situations could conflict with students' natural sense making about what should constitute linearity. In addition, students who work with contrived data will lack the opportunity to directly explore the nature of the relationship, making sense of why constant ratio is an appropriate mathematical model for the phenomenon in question. Similarly, tasks presenting approximately linear data, either due to measurement error or the inexact nature of the phenomenon, may prevent students from isolating the importance of ratio relationships. While inexact or approximately linear data are fully appropriate data to investigate, particularly in terms of highlighting the power of mathematical models for making sense of messy data, these contexts should occur after students have already formed an understanding of linearity as a constant rate of change.

Teachers who understand the importance of experience and quantitative reasoning can help their students focus attention on quantities and the language of quantitative relationships. Although students might be likely to attend to number patterns alone once they are extracted from the situation and represented in tables or other forms, teachers can intervene to draw students' attention back towards the quantitative referents of numbers and patterns. In addition, teachers can incorporate the language of quantities into the classroom discussion; for example, when students are likely to describe patterns in a table such as "each time it goes by 5, it goes up by 2," a teacher could ask students to describe whether the clown walks the same distance for a given amount of time throughout the table.

One important feature of this study's results is that neither group of students focused exclusively on quantities and quantitative relationships, even when they worked within quantitatively rich contexts. This serves as an important reminder for those concerned with instructional implications: real-world situations will not serve as a panacea. Students may focus on any number of features in a problem situation, and this focus may not always include relationships between quantities. The problem situations presented to students are less important than students' interpretations of these situations. Therefore teachers play an important role in shaping a classroom discussion, posing appropriate questions, inserting new information, and otherwise guiding students to think carefully about quantitative relationships. If students are continually encouraged to develop emergent quantities, they will be poised to produce generalizations that are correct, powerful, and well connected to other knowledge.

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