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Hidden lessons: How a focus on slope-like properties of quadratic functions encouraged unexpected generalizations☆

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ABSTRACT

This article presents secondary students' generalizations about the connections between algebraic and graphical representations of quadratic functions, focusing specifically on the roles of the parameters a, b, and c in the general form of a quadratic function, $y = ax^2 + bx + c$. Students' generalizations about these connections led to a surprising finding: two-thirds of the students interviewed identified the parameter a as the "slope" of the parabola. Analysis of qualitative data from interviews and classroom observations led to the development of three *focusing phenomena* in the classroom environment that inadvertently supported a focus on slope-like properties of quadratic functions: (a) the use of linear analogies, (b) the rise over run method, and (c) viewing a as dynamic rather than static.

Many mathematics educators emphasize the value of real-world investigations and applications for helping students develop a more robust understanding of functions (Arnold, 2006; Borenstein, 1997; Hershkowitz & Schwarz, 1997; Schorr, 2003). In response to this emphasis, several curriculum development projects now include student-centered investigations and activities that capitalize on the use of realistic data and scenarios. This article presents students' generalizations about quadratic functions after participating in one such investigations-focused unit from Discovering Advanced Algebra (DAA, Murdock, Kamischke, & Kamischke, 2004).

Although our initial intent was to examine how students generalized from the real-world situations they encountered in the classroom to novel tasks about quadratic growth, the classroom emphasis ultimately settled on connections between algebraic and graphical representations of quadratic functions. Students' generalizations about these connections led to a surprising finding: two-thirds of the interview sample described the role of the parameter a in $y = ax^2 + bx + c$ as the "slope" of a quadratic function. Given this finding, we returned to the classroom data in order to identify potential sources for the students' generalizations. Results of our analysis revealed three focusing phenomena (Lobato, Ellis, & Muñoz, 2003) that directed students' attention towards slope-like properties of parabolas. In this article we (a) present students' generalizations about the roles of the parameters a, b, and c in the general form of a quadratic function, (b) discuss potential sources of those generalizations, and (c) identify the three focusing phenomena that encouraged a view of a as the slope, discussing their role in supporting students' incorrect generalizations.

1. Making sense of algebraic and graphical representations of quadratic functions

Quadratic functions served as an appropriate domain both for its extension beyond linear functions, thereby providing a site for studying students' generalizations across families of functions, and for its relative lack of attention in the literature

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compared to work on linear functions. Studies focusing on quadratic functions have mainly detailed students' difficulties in a few key areas, including (a) connections between algebraic, tabular, and graphical representations, (b) a view of graphs as whole objects, (c) struggles to correctly interpret the role of parameters, and (d) a tendency to incorrectly generalize from linear functions. Researchers argue that in order to develop true function understanding, students must move flexibly between representations (Dreyfus, Hershkowitz, & Schwarz, 1997; Moschkovich, Schoenfeld, & Arcavi, 1993). Work examining mathematical flexibility (e.g., Bulgar, Schorr, & Warner, 2004) has found that student knowledge is more flexible and durable when learned in conditions that encourage mathematical understanding through activities such as model making. For instance, Walter and Gerson (2007) detailed learners' abilities to make sense of slope using concrete models, which enabled an alternate understanding of slope as repeated addition. Studies examining mathematical flexibility with quadratic functions, however, reveals students' difficulty connecting algebraic representations of functions with graphical aspects, and in translating between algebraic and graphical settings (Bussi & Mariotti, 1999; Leinhardt, Zaslavsky, & Stein, 1990). As Dreyfus and Halevi (1991) remind us, "One of the central difficulties for students in the process of constructing their mental image of function is the establishment of the connection between the formula defining a function algebraically and its graphical representation." (p. 44)

Similarly, studies examining students' conceptions of graphs of parabolas have documented difficulties in making sense of shifts and translations (Zazkis, Liljedahl, & Gadowsky, 2003), identifying non-visible aspects of figures such as y-intercepts (Zaslavsky, 1997), and the tendency to treat graphs as whole objects, or pictures, rather than as a collection of individual points (Moschkovich et al., 1993; Santos, 2000). Research examining an understanding of the roles played by the parameters in functions such as $y = ax^2 + bx + c$ has shown that students can generally articulate a as indicating the "opening" of a parabola, but struggle to provide further detail (Dreyfus & Halevi, 1991). Generalizing from an understanding of quadratic equations, students have also stated that the value of a can be chosen arbitrarily: for example, while $y = x^2 + 2x - 3$ and $y = 2x^2 + 4x - 6$ are different functions, as equations set equal to 0, they would have the same roots (Zaslavsky, 1997). Given difficulties connecting algebraic and graphical representations, it is not surprising that students may struggle with describing the effect that changing the parameters a, b, and c will have on a graphical representation.

A great deal of the quadratic functions literature has also documented students' tendencies to generalize from linearity (Chazan, 2006; Buck, 1995; Schwarz & Hershkowitz, 1999). For instance, Hershkowitz and Schwarz (1997) found that students often use linear interpolation and extrapolation, as opposed to non-linear inter/extrapolation or reasoning from formal definitions, even when it produces nonsensical results. Zaslavsky (1997) documented cases of students using two points to calculate a slope when trying to find the equation of a quadratic function, and suggested that an extensive investigation of linear functions prior to studying quadratic functions could impede understanding due to inappropriate generalization from linearity. These results were in line with our own finding that students interpreted the value of the parameter a as the slope of the parabola, leading us to hypothesize initially that students may have simply generalized from their linear function understanding. However, further investigation revealed phenomena in the classroom that inadvertently supported connections to slope, suggesting that features of classroom environments may encourage generalizations from linearity in ways unintended by teachers or students.

2. Generalization and focusing phenomena

Although research investigating students' tendencies to incorrectly generalize from linear functions typically employs the term "overgeneralization", we avoid that language because it is inconsistent with our views of generalization. Specifically, we apply the actor-oriented transfer perspective (Lobato, 2003) to generalization, which advocates for a shift from the observer's orientation to the actor's orientation when studying processes of transfer or generalization. Under this framework, the researcher abandons normative notions of what should count as transfer (or correct generalization) and instead seeks to understand the processes by which learners generate their own relations of similarity, regardless of their correctness. This allows the knowledge that is generalized to become an object of investigation, and frees the researcher to focus on what appear to be salient features of similarity from the student's point of view.

Generalization, then, is not restricted to a formal verbal or algebraic description of a correct rule, but instead is sought by examining the similarities, connections, and extensions that *students* perceive to be general. In the spirit of Kaput's (1999) view, we therefore define generalization as engaging in at least one of three activities: (a) identifying commonality across cases, (b) extending one's reasoning beyond the range in which it originated, or (c) deriving broader results from particular cases (Ellis, 2007). The students who identified the role of the parameter a as the slope may indeed have been generalizing from their understanding of linear functions; we would characterize this connection, if it is articulated, as a generalization rather than an overgeneralization, because the student herself would not perceive it as an overgeneralization.

The notion of focusing phenomena is a researcher's construct developed in an earlier study (Lobato et al., 2003) as a way to account for students' unanticipated generalizations. Focusing phenomena are observable features of classroom environments that regularly direct students' attention towards particular mathematical properties over others. They can occur as a result of not only a teacher's behavior, language, and gesture, but also through students' language and gesture, features of curricular materials, and the ways in which students and teachers use artifacts such as concrete manipulatives or graphing calculators. In this sense, focusing phenomena are co-constructed phenomena, and usually emerge through an interaction between students, teachers, textbooks, artifacts, and other features of classrooms environments. The construct is designed to help researchers account for the multiple agents that work together to direct students' mathematical attention towards certain properties over others. While actor-oriented transfer is a construct that allows researchers to attempt to identify generalizations from the student's perspective, focusing phenomena is a construct that enables researchers to identify potential mechanisms driving the generalizations identified. In this article, we present three focusing phenomena as a potential explanation for the unanticipated generalizations that the students produced about the roles of the parameters in $y = ax^2 + bx + c$.

3. Methods

The data reported in this article were gathered at an urban high school in a midwestern city in the United States. Of the school's 1839 students, the majority (57.5%) are White and the remaining students are African American (22.8%), Hispanic (8.1%), Southeast Asian (7.6%), Other Asian (3.5%), and Native American (0.5%). Nearly 13% of students are classified as ESL and 42% are classified as free/reduced lunch. The school was selected because of its diverse student population, the mathematics department's stated interest in supporting standards-based mathematics, and the eagerness of the cooperating teacher to participate in the study.

3.1. Participants and data collection

This study consisted of a series of videotaped classroom observations during a quadratic functions unit and two sets of individual interviews with 8 students. The collaborating teacher, Ms. R, was recruited for the study based on her stated commitment to the NCTM Principles (2000) and her interest in the project. Ms. R had been teaching since 1993 and described herself as "deeply passionate about social justice issues and the use of mathematics as a lever to effect change for individuals and groups" and "having the goal of fostering all students to see themselves as doers and creators of mathematics." Classroom observations took place in her Algebra II/Trigonometry class, which consisted of 34 students. Ms. R was the regular classroom teacher and observations occurred during one unit in her class. The normal text for the classroom was DAA. We collected video data on 16 days comprising an instructional unit on quadratic equations, and also collected written artifacts in the form of tests, quizzes, and homework assignments. One camera was placed in the back of the classroom with the primary aim of capturing the whole-class discussions, investigative activities, and group work. One researcher took field notes and operated the camera. During roughly half of the class sessions, a second researcher was also present. We attempted to avoid disrupting the normal flow of the class, reserving closer investigation for the interviews.

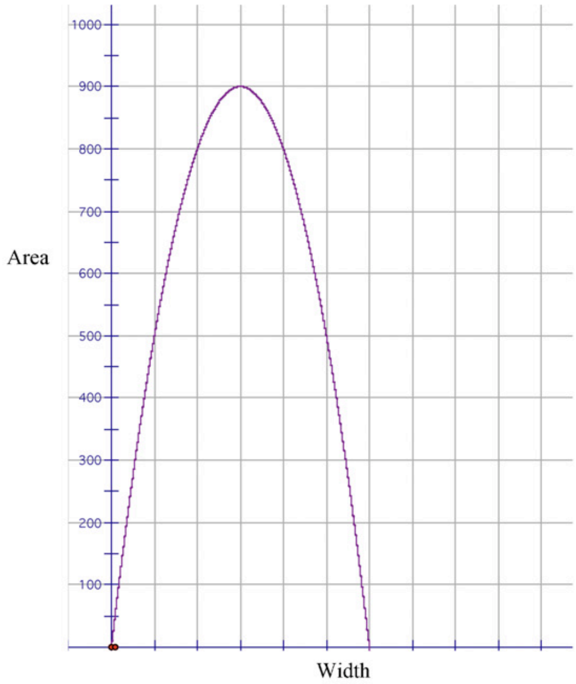
Two sets of semi-structured interviews (Bernard, 1988) provided insight into the students' generalizations about the nature of quadratic growth, the meanings of the quadratic equations presented in class, the characteristics of quadratic graphs, and the relationships between quadratic data presented in tabular, equation, and graphical forms. The first interview took place approximately one week into the instructional unit, and the second interview occurred approximately one week later. Each videotaped interview lasted about 50 min. We made efforts to ensure that the students were not pressured to come up with the "right" answers but, instead, were encouraged to explain what they were thinking. Transcripts of the interviews captured not only the students' utterances, but also their written explanations, drawings, and gestures. Descriptions of students' gestures and drawings served as an additional source of information about students' reasoning.

Eight students participated in each interview. The interviewees were identified by the teacher as those who (a) had grades in the A–C range, (b) were particularly articulate and capable of describing their ideas, (c) indicated an interest in participating in the project, and (d) demonstrated good class attendance and participation. Because many of the classroom members appeared to be insufficiently prepared to make sense of the DAA curriculum, we selected interviewees with the above qualities in the hopes that they would have better-developed mathematical ideas to share during the interviews. Their teacher identified the interviewees as those who were particularly successful on their homework assignments, quizzes, and tests. Two of the students were African American, 1 student was Latina, 1 student was Asian American, and 4 students were White. Four students were male and four were female, and 1 student was an English Language Learner. We have used gender-preserving pseudonyms for all students.

3.2. Interview instruments

The interviewers presented students with a series of questions about quadratic function concepts. As the intention of the interviews was to identify student generalizations regarding quadratic functions, we did not draw the questions directly from the DAA text or from classroom tasks. Instead, we designed tasks that would elicit students' ideas about (a) the nature of quadratic growth and what constitutes data that count as quadratic, (b) the connections between data in tabular, graphical, and equation form, and (c) which parts of quadratic graphs and equations are the most important to students, and what roles they play. Two sample problems from the interviews are shown below. We used the rabbit pen problem (see Fig. 1) to determine students' understanding of how a quadratic function's coefficients influence its shape and to examine the types of equations that students viewed as appropriate for the graph. The parabola drawing task (Fig. 2) elicited students' generalizations about what quadratic functions look like and how their appearance is shaped by the various parameters in the associated equations. The analysis presented in this article is part of a larger study on students' generalizations about quadratic relationships.

Ravi has 120 m of fence to make a rectangular rabbit pen. He wants to enclose the largest possible rectangular area. Here is Ravi's graph of the relationship between width and area:



A. Making sense of the points:

a) What does the graph suggest about the area of the pen?
b) What does this point mean in the situation? [Point to (10,500)].
c) How big do you think the area of the pen will be when the width is 50 m? 70m?
d) One of the pens Ravi made was 756 square meters. What might the width of the pen be?
e) What does this point (point to the vertex) tell you about the situation?
f) What does this point (point to the x-intercept) tell you about the situation?

B. Quadratic equation:

a) Can you write an equation or formula for the graph?
b) How do the numbers in the equation correspond to the graph and the situation? For example, "Does the 60 have any meaning in this graph? Does it mean anything in the relationship between width and area?"
c) Someone else I interviewed came up with an equation of $y = -x^2 + 60x$. Do you think this equation might be correct? [Present an erroneous equation to student in case they come up with a correct one]
d) Is the number 60 related to the graph or situation in any way?
e) Why is the x^2 term negative? Does this have meaning for the graph or situation?
f) What does the x mean in this situation?
g) What does the y mean in the situation?

Fig. 1. The rabbit pen task.

3.3. *The quadratic functions instructional unit*

Classroom observations spanned the entire instructional unit (see Fig. 3) for DAA's seventh chapter, "Quadratic and Other Polynomial Functions". The cooperating teacher employed real-world investigation tasks to support (a) fluency between general, vertex, and factored forms of quadratic equations; (b) writing equations; and (c) solving equations using the quadratic formula or the "quadratic regression" program on the TI-83 calculator. (The quadratic regression program used three (x, y) pairs in order to generate values for a, b, and c for the general form $y = ax^2 + bx + c$.)

The DAA approach relies on investigations to develop mathematical ideas. For example, on day 3 of the unit, students began a 2-day investigation of the speed at which a can will roll down an inclined plane. Students collected data with a

Please draw a quadratic function in the box below:

a) What makes this graph quadratic?
b) Which parts of this graph are important or special? Why?
c) Can you give an equation for your graph? (Prompt for a best guess if the student feels unable to do so).
d) Probe the student to explain how the parameters in the equation he or she writes (for example, the a, b, and *c* in y = ax^2 + bx + c) affect the graph.

Ask how each of the parameters affect the special parts of the graph mentioned earlier.

Fig. 2. The parabola drawing task.

motion detector attached to their graphing calculator, and recorded various position-time points. Using tabular and graphical representations of the can's position with respect to time enabled students to explore the (approximately) quadratic relationship between distance and time. The students used their graphing calculators to create graphs of the data, and then used the calculator's quadratic regression program to find equations in the general form $y = ax^2 + bx + c$. This investigation allowed the students to observe the algebraic and graphical appearance of quadratic functions in advance of the introduction and definition of terms such as *vertex* or *stretch factor*.

Although the textbook lessons emphasized group work and real-world investigations, these investigations were often more prescriptive than exploratory due to their step-by-step nature. The teacher used a variety of instructional formats, such as whole-class discussion and small groups, but direct instruction was also a substantial part of the course. This dichotomy between group-based investigations and prescriptive, repetitive exercises was reflected both in the teacher's instruction and in the textbook, which the cooperating teacher described as "reform lite." Although the text provided opportunities for rich and thoughtful investigations into realistic phenomena, these opportunities did not always manifest themselves in the students' day-to-day experiences in the classroom.

3.4. Data analysis

Data analysis began with the coding of the interview transcripts. Our goal was to develop categories of students' generalizations about the nature of quadratic growth, the critical parts of quadratic graphs, and the meanings of the parameters in quadratic equations. This article reports on the analysis of this third set of generalizations, students' generalizations about the parameters, specifically the roles played by the parameters a, b, and c in the general form $y = ax^2 + bx + c$. The cooperating teacher's emphasis on this general form, in combination with the students' use of the quadratic regression calculator program, meant that the students focused heavily on the general form over other forms of a quadratic equation, such as $y = a(x - h)^2 + k$. However, because the teacher emphasized that the parameter a played the same role across all forms of quadratic equations, occasionally students discussed a in forms other than the general form.

Interview coding followed the interpretive technique in which the categories of types of generalizations were induced from the data (Glaser & Strauss, 1967; Strauss & Corbin, 1990). We coded Interview I first in order to develop tentative

	Day	Topics	
Week 1	1	What does a quadratic look like? What does it mean?	
	2	Introduction of the general form, finite differences method, quadratic regression, matrix method. Investigation: diagonals of polygons.	
	3	Introduction of vertex form and factored form, FOIL	
	4	<no instruction – test day>	
	5	Investigation: rolling objects down inclines. Converting between the three forms, role of a & c, a as "stretch factor"	
Spring break			
Week 2	6	Using the three forms to make sense of data and describe graphs	
	7	Using the three forms, completing the square	Interview 1
	8	Using the calculator to find the vertex	
	9	Completing the square	
	10	Quadratic formula, using calculator's "intersect" function	
Week 3	11	Non-real roots, the perils of calculator approximations, investigation: dropping objects	
	12	Projectile motion, making equations from real situations, the meaning of roots of negative numbers	Interview 2
	13	Numbers: complex, real, imaginary; powers of i, computations (+-x/) with complex numbers	
	14	Complex numbers, factoring polynomials, roots vs degree of a polynomial	
	15	Roots versus degree of a polynomial	
Week 4	16	Factoring polynomials	
	17	Review, projectile motion	
	18	<no instruction – test day>	

Fig. 3. Outline of the quadratic instructional unit.

codes for students' generalizations about a, b, and c. Once a common set of categories emerged and stabilized, we then applied the categories to the data from Interview 2. Based on the Interview 2 data, we made appropriate modifications to the categories as necessary. We then applied the modified categories to Interview I until the data had been saturated.

Once categories of students' generalizations had stabilized, we turned to the classroom videotapes, written artifacts, and the textbook in order to determine the focusing phenomena for students' generalizations. In the majority of the cases, the students' generalizations were easily connected to the classroom environment through an examination of the teacher's and the students' language, the emphasis in the text, or the students' use of the graphing calculator. However, there was a particular category of generalizations about the parameter a that we did not anticipate, namely that a was related to the "slope" of the parabola. In an attempt to understand how these generalizations might have occurred, we turned our analytic focus towards the instructional environment, searching for potential supports for those generalizations. We searched for regularities in the classroom environment that focused students' attention on particular mathematical properties of a over others, particularly features that focused attention on slope-like properties of a.

By searching for ways in which a might have been linked to ideas about slope, we compiled all of the video episodes that demonstrated any observed features that appeared to focus attention on a connection between the role of a and the properties of slope. Once these episodes had been located, we then created and coded categories for the different features that appeared to focus students' attention to slope in relationship to a. These features represent a researcher's understanding of the phenomena that occurred in the classroom, and the categories represent possible explanations for students' generalizations about a and slope. Three categories of focusing phenomena emerged, and we developed collections of the coded episode clips that related to the three potential focusing phenomena. Codes for these episodes were revised against repeated analysis passes in an iterative process. In this manner, the classroom data analysis also relied on the constant comparative method used in the development of grounded theory (Strauss & Corbin, 1990).

4. Results

We present the results in two sections. In Part I, we present students' generalizations about the parameters a, b, and c in the general form of the quadratic function, $y = ax^2 + bx + c$. Students' generalizations are explained and connected to features of their instructional environment in every case except one, in which students described a (and in two cases, b) as the "slope" of the parabola. Due to the unexpected nature of students' beliefs that those parameters represented slope, we then present in Part II an analysis of the mechanisms behind this category of generalizations. We describe three focusing phenomena that directed students' attention towards slope-like properties in relationship to the parameter a, and close with a discussion of the ways in which certain focusing phenomena can compete with more direct and explicit foci in the classroom to encourage unexpected generalizations.

Table 1
Students' generalizations of *a*, *b*, and *c*.

	Alexis	Andrew	Amata	Corine	Jack	Kordell	Mandy	Tashi
c is the *y*-intercept	√	√	√	√	√	√	√	√
c is the *x*-intercept				√	√			
c translates the graph	√	√	√		√			√
c affects the vertex					√			
b affects the vertex	√	√	√	√	√	√	√	
b translates the graph		√	√		√	√		√
b is the width			√	√				√
b is the *x*-intercept				√	√			
b is the slope	√					√		
a affects the shape	√	√	√	√	√	√	√	√
a is the stretch factor	√			√	√	√	√	
a determines up or down	√		√		√	√		√
a is the slope	√	√	√	√		√		√

4.1. *Part I: Students' generalizations about the parameters a, b, and c*

The 8 students' generalizations about each of the three parameters are shown in Table 1. We discuss interpretations of each of the parameters in turn.

4.2. *Interpretations of c*

Students' generalizations about the role of the parameter *c* predominantly fell into four categories: (1) *c* is the *y*-intercept, (2) *c* is the *x*-intercept, (3) *c* translates the graph, and (4) *c* affects the vertex or is part of the vertex. Every student generalized the meaning of *c* in two ways except Jack, who demonstrated all four interpretations of *c*. The most common interpretations of *c* were as the *y*-intercept, and as translating the graph.

4.2.1. *c is the y-intercept*

All 8 interviewees stated, at some point, that *c* was the *y*-intercept. Their explanations frequently relied on assertions of equivalence: for example, Andrew explained, "that's [points to the "−1000" in $y=(-1/2)x^2+60x-1000$] supposed to be the *y*-intercept". Some students, however, used language of moving the *y*-intercept, as when Alexis noted that *c* "raises or lowers the *y*-intercept". The movement language bore a resemblance to the generalization that *c* translated the entire graph. The ubiquity of the *y*-intercept generalization across the entire interviewee group is likely attributable to the frequency with which the teacher emphasized the role of *c* as the *y*-intercept:

Ms. R: *b* is just kind of a coefficient, but *c* is very useful to us. Do you remember what *c* tells us?
Amata: The constant?
Ms. R: *c* is the constant term, and where is it on the graph?
Amata: Oh, isn't it where it starts from the *y*?
Ms. R: Yes! That's the *y*-intercept, and if you look at the equation, when *y*, when you're on the *y*-intercept, what's your *x*-value? You're on the *y*-axis, what's your *x*-value?
Students: 0.
Ms. R: 0. What happens if the *x*'s are 0, what's left? *c*, so that's your *y*-intercept.

The participating teacher explicitly discussed the role of *c* 13 times during the quadratic unit, and each time she emphasized that it was the *y*-intercept.

4.2.2. *c is the x-intercept*

Two of the 8 interviewees also made use of the generalization that *c* was the *x*-intercept. When asked how *c* changes the appearance of the graph, Jack said that it does "nothing really to the shape of the graph. It'll just stay where—where, like, it goes through the *x*-axis or whatever." Corine, meanwhile, stated that the −18 term in the equation $y=x^2-7x-18$ was "the other point (root)". This conception may be attributable to confusion over whether *c* affected the *x*- or *y*-intercept: although the teacher always referred to *c* as the *y*-intercept, the students occasionally referred to *c* as "the intercept."

4.2.3. *c translates the graph*

Five interviewees made use of the generalization that *c* translates the graph vertically. Amata, for example, explained that "[i]f I were to add 9, the graph will go up [shows calculator to interviewer while making an 'up' gesture over the screen] 9." The students may have connected the role of *c* to a vertical translation because a prior unit had emphasized vertical and horizontal translations of graphs. In fact, the DAA text used the quadratic family and the prototypical parabola $y=x^2$ to illustrate how to translate graphs: "If you translate the graph of $y = x^2$ horizontally h units and vertically k units, then the equation of the translated parabola is $y = (x - h)^2 + k$." (p. 196)

4.2.4. *c affects the vertex or is part of the vertex*

Finally, 1 student claimed that c affected the vertex or was part of the vertex. Jack described the coordinates for the vertex, saying, "b – and then isn't it like you take, it ends up being a, like – b comma c [writes $(-b, c)$]. That's, like, the vertex of it?" It is possible that Jack confused $(-b, c)$ with $(-h, k)$ in the vertex form $y = a(x + h)^2 + k$.

4.3. *Interpretations of b*

Students' generalizations about the role b played in $y = ax^2 + bx + c$ were not characterized by any significant trends across the interview participants. The students expressed a number of different hypotheses about b's role, including (1) b affects the vertex, (2) b translates the graph horizontally, (3) b is the "width" of the parabola, (4) b is the x-intercept of the graph, and (5) b is the slope of the graph. Every student except Mandy demonstrated at least two interpretations of b, and the most common interpretation was that b affects the vertex. The reason for the large number of generalizations about the role of b and the lack of clear defining trends is likely due to the absence of any discussion of the role played by b, either in the text or in the classroom discussions. The parameter b only came up in discussion twice during the unit. Once, on Day 5, the teacher remarked that b was "just kind of a coefficient" and then moved on to the discussion of c. On Day 15, Kordell asked Ms. R what b did, and she explained, "it doesn't really do much of anything, other than when you use the quadratic formula." We will describe each of the five conceptions for b briefly, focusing most on the fifth conception, b as the slope of the graph.

4.3.1. *b affects the vertex*

Seven participants described b as affecting the vertex in some form, but no student could articulate how. A typical response describing the role of b was Mandy's: "I don't know, I can't decide. Because now it's still the vertex. . .". Jack explained, "I think, I think that determined like, the, I don't know. . .I think it determines like the vertex, but I'm not sure." The uncertainty in both Jack and Mandy's responses typified the students' confusion over the role played by b. A generalization that b was connected to the vertex might have occurred because for the general quadratic form $y = ax^2 + bx + c$, the teacher and the text both articulated the roles of a and c in ways that were not connected to the vertex; a affected the shape of the graph, and c was the y-intercept. Given the students' focus on the vertex as an important part of a parabola, they might have determined that the remaining parameter b must affect the vertex, but in ways that they could not clearly articulate.

4.3.2. *b translates the graph horizontally*

Five students thought that b might result in a horizontal translation. For instance, Kordell suggested "I guess this just moves it [the graph] over to the right," while Tashi explained "if you add a number here [points to b in $y = ax^2 + bx + c$], it'd either make it move like, if it's like three like this, then it would go that way [draws an arrow to the left] and if it was negative three it'd go that way [draws an arrow to the right]." It is possible that the students were demonstrating transfer between their understanding of h in the vertex form, $y = (x - h)^2 + k$, and the role of b in the general form. Because the role of b was unspecified, the students might have searched for an understanding related to their knowledge of parameters in other forms of equations; the parameters in the vertex and factored forms either determined the vertex of the parabola, its roots, or vertical or horizontal translations.

4.3.3. *b is the width of the parabola*

Three students described b as affecting the "width" of the parabola, either in terms of the distance between the x-intercepts or the distance between two arbitrarily determined points on the parabola with the same y-values. For instance, in the first interview Amata described the role b played in a parabola with x-intercepts at 0 and 60: "b is the width between, like it's how the distance between 0 [$x = 0$, the leftmost root] and 60 [$x = 60$, the rightmost root], which is 60." Generally describing her idea, she explained, "it kind of like shows how like if it's like a wide parabola like this [gesturing with her arms] or if it's a skinny parabola." Because the equation for the rabbit pen task was $y = -x^2 + 60x$, it is possible that the students saw that 60 happened to be both the value for b and the distance between the two roots.

In the second interview, Amata was given the parabola drawing task (Fig. 2) and asked to draw a parabola and identify which parts were important. She responded, "How wide, and where it crosses the y-intercept." When asked where the width would be measured, Amata produced the following drawing (Fig. 4) and explained, "from the farthest points, I would think, like where it starts to stay constant, almost." After a moment of introspection, Amata said, "Well, I don't know. . .I think any spot would be appropriate, like any two points on either side." Prompted to find an equation for the parabola, Amata choose some points from the parabola, typed them into her calculator, ran the "quad regression" program, and got $y = x^2 + 1x - 1$ as an equation for the graph. When asked about the role of the 1 in $1x$, she again described it as determining a parabola's width, which was an idea she had first articulated in the first interview. It is possible that Amata may have made connections between the role of a, which was often described as affecting the width of the graph, and the role of b.

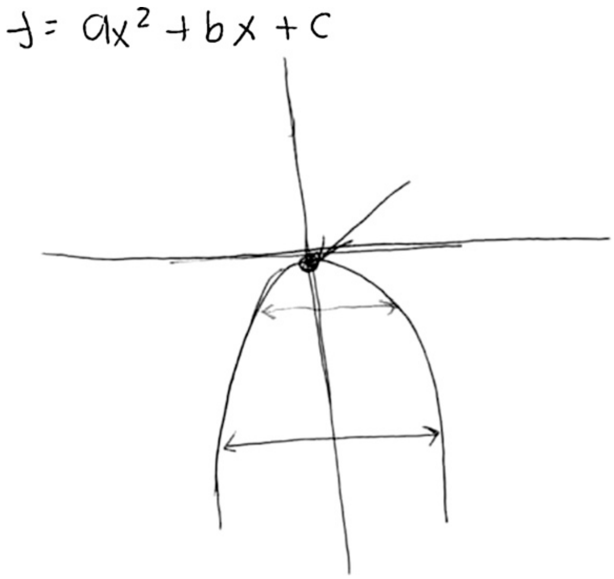


Fig. 4. Amata's drawing of the important parts of a parabola.

4.3.4. b is the x-intercept

Two students identified b as the x-intercept. When asked what the 60 meant in $y = -x^2 + 60x$, a hypothetical student's equation for the rabbit pen task, Jack replied, "I think it's the x-intercept." This was a reasonable answer given that the graph he looked at did indeed have 60 as an x-intercept. Similarly, Corine studied a graph that appeared to have x-intercepts of approximately -7 and 20. When shown a hypothetical student's equation for the graph, $y = x^2 - 7x - 18$, she noted that -7 would be a root: "it'll be one of these points [pointing to the roots]."

4.3.5. b is the slope of the graph

Two students described b as the slope of the graph. When asked what the 60 meant in $y = -x^2 + 60x$, Kordell answered "I guess it's the slope." Alexis also indicated that 60 would be the slope of the graph. In interview 2, when given the parabola drawing task, Alexis drew a parabola and ultimately wrote the equation $y = x^2 + 3x - 8$. When asked how she chose the 3, Alexis responded "because it doesn't really have that big of a slope, and it doesn't really have a really small slope." When asked if there was a particular part of the graph that she looked at in order to judge the slope, however, Alexis responded that she "just looked at it as a whole."

The connection to slope was surprising given that the topic was quadratic functions rather than linear functions, and parabolas do not have a constant slope. While Kordell and Alexis' ideas about the role of b may have been a fluke, the participants' responses about the role of a indicated further connections to slope: 75% of the participants indicated that a represented the slope of the graph at some point in the interviews. This was one of four generalizations about the role of a; all four are discussed below.

4.4. Interpretations of a

The interview participants demonstrated four interpretations of a, three that were correct in the eyes of their teacher, and a fourth that was connected to slope. The students generalized that the parameter a: (1) affects the shape of the parabola, (2) is the "stretch factor", (3) determines whether the parabola faces up or down, and (4) is the slope of the parabola. Two students, Alexis and Kordell, demonstrated all four interpretations of a. Every other student demonstrated at least two interpretations, which is not surprising given the variety of ways in which the teacher discussed the role of a in the classroom. In this section we will present the students' generalizations and then include data supporting the origins of these generalizations for the first three categories. Slope will be discussed separately in Part II.

4.4.1. a affects the shape of the parabola

All 8 participants stated that a affected the shape of the graph by making it wider or narrower. For instance, Alexis described the parameter's effect on the graph by explaining that a "makes it wider or skinnier." Similarly, Jack wrote the equation "$y = .25(x - 4)^2 - 8$" to describe a parabola he had drawn. The interviewer asked him what would happen if .25 were changed to 2, and he responded, "then it would look. . .then it would be, like, thinner. Like this [drawing a skinny parabola narrower than the original parabola, but with the same vertex]." He then explained his general rule: "the smaller the number that you're multiplying gets, the wider it gets." In limited cases, students viewed a as affecting vertical, rather than horizontal, qualities of the graph, as Tashi did when explaining what would happen if $-x^2$ were changed to $-2x^2$: "I guess it would just go up faster. Like, I mean, like it would go higher, it would double this height."

Students preferred to view a as changing the width, rather than the height, of the parabola. This preference is displayed by Alexis as she uses "larger" to say "wider":

Int: Can you come up with an equation for this graph [see Fig. 10]?
Alexis: [Writes $y = (-1/2)x^2 + 3x + 5$].
Int: So you have $(-1/2)x^2 + 3x + 5$. Okay. Why'd you choose $-1/2$?
Alexis: Because it has to be negative because it's upside-down.
Int: Mm hmm.
Alexis: And 1/2 because it's larger than a normal parabola.
Int: Oh, okay. Larger in what way?
Alexis: It's wider.

Mandy makes a similar choice in explaining the role of a when discussing her equation, $y = (5/6)x^2 - (1/6)x + 1$:

Int: Now, in this graph, I see you have $(5/6)x^2$. So what, what's the 5/6 doing?
Mandy: Um, making the graph shorter, or more wide.

. . .

Int: So is it shorter or wider, or are those the same, or. . .?
Mandy: Wider. Just because it gets wider, it looks shorter.

Although the students indicated that a influenced the shape of the parabola, they did not demonstrate a belief that changing the value of a would change the location of its vertex. While this view is incorrect – changing a does change the location of the vertex when b is not zero – it is also reasonable given the focusing phenomena present in the classroom.

In particular, the uniformity of student beliefs about the value of a as influencing the shape of the parabola but not the vertex is unsurprising given the instructional emphasis on this role for a. DAA referred to the value of a as the "vertical scale factor" (p. 368). In a prior chapter, students had learned about translations, stretches, and shrinks of graphs. The coefficient a for any function $y = af(x)$ was described as "a vertical stretch or shrink by a factor of a. When $a > 1$, it is a stretch; when $0 < a < 1$, it is a shrink." (p. 213) Given this introduction for any function, the discussion of the role of a did not include any mention of the possibility that a would change the location of a vertex.

Although the text's language emphasized the role of a as influencing the graph vertically, the teacher's use of gesture, language, and problem choice emphasized a focus on the role of a as influencing the *width* of the parabola. For instance, a week into the unit, Ms. R demonstrated how to graph an equation in which a was 1/2:

Ms. R: Which way does it [the parabola] open, up or down?
Students: Up.
Ms. R: Up?
Students: It's up.
Ms. R: And, is it–
Student: Smiley face.
Ms. R: And is it tall and skinny or short and wide?
Students: Short and wide.
Ms. R: Short and wide.

Ms. R's gestures particularly focused attention on a's value in determining the width of the parabola, rather than influencing the height or the location of the vertex:

Ms. R: Just as Alex asked the question before, what's a? Always the lead coefficient in general form, so you can pull it from that. It's the stretch factor.
Amata: What's that?
Ms. R: This is a regular parabola [holds her arms up in a parabolic shape], this is a parabola with a stretch factor of 2 [moves her arms up in order to demonstrate a narrower parabola], this is a parabola with a stretch factor of 1/2 [moves her arms down to demonstrate a wide, flat parabola].

By using her arms to demonstrate how a would affect the graph, Ms. R could have contributed to a generalization that changing a would leave the vertex the same. Moving her arms to indicate a wide or narrow parabola did not allow the students to perceive any effect on the vertex; the vertex was centered in her body and remained stable as her arms indicated a parabola's changing shape. The teacher's gestures, then, depicted a scenario in which changing the value of a changed the shape, but not the location, of the parabola. Although this depiction of the role a plays is accurate for parabolas in the families

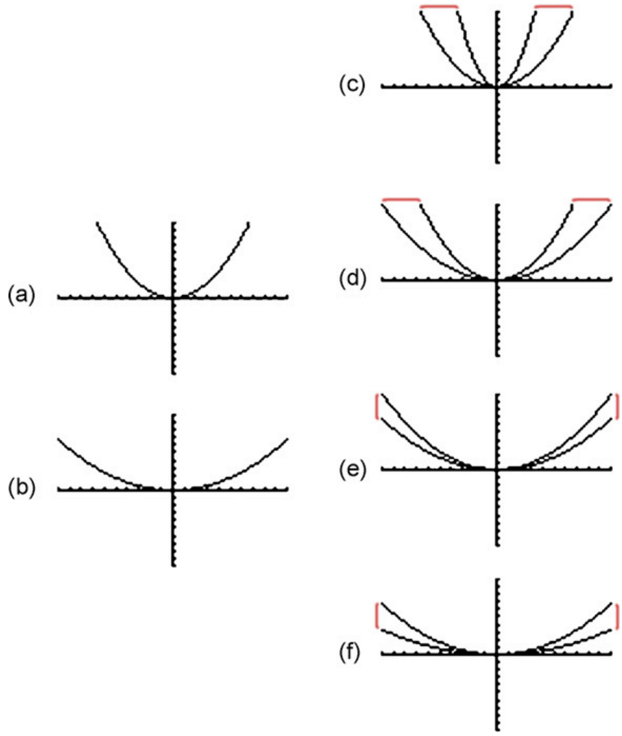


Fig. 5. The effect of the calculator window on perceptions of "wider" versus "taller" when the parameter a is varied.

$y = ax^2$ or $y = ax^2 + c$, the students generalized this idea of a affecting the width but not the vertex to all parabolas in the form $y = ax^2 + bx + c$.

The students mirrored this gesture when describing the role played by a to the interviewer; they used their hands, their arms, and their fingers to indicate wider and narrower parabolas. For instance, when describing the influence of a for the equation $y = .5x^2 + x - 3$, Tashi explained, "Well, I mean, .5 like, if you move down to a fraction in front of the a, it makes it like more of a like, like in decimals I guess, but like make it, make the parabola open up more [makes a 'wider' gesture with his hands]."

The use of calculators may also have played a role in the formation of generalizations regarding a's role. Students used their calculators to explore the effect of varying parameters by graphing a prototype function (e.g., $y = x^2$) and the same function with a changed parameter (e.g., setting $a = 2$ to get $y = 2x^2$) and then comparing the two graphs. One problematic feature of graphing calculators, however, is that most parabolas will reach the vertical boundary of the calculator window before they reach the horizontal boundary (see Fig. 5). When graph (a) [$y = 0.225x^2$] is changed to reflect larger or smaller values for a, the changed graphs appear skinnier or narrower in the standard calculator window, as seen in graphs (c) and (d). This visual display is more common than graph (b), which is the graph of $y = 0.0667x^2$. In that case, changing the value of a very slightly will make the parabola appear taller or shorter, as seen in graphs (e) and (f). However, the value of a must be less than 0.1 in the standard window for a typical $y = ax^2$ parabola to reach the horizontal boundary rather than the vertical boundary, and the majority of the parabolas the students tested had a values larger than 0.1. Therefore most changes in a will appear to control how "wide" or "skinny" a parabola will be rather than how "tall" or "short" it is. The nature of the calculator window in effect focused students' attention on the parabola's changing width rather than its changing height. This feature of the calculator window, coupled with Ms. R's gestural explanations of a's role, reinforced the notion that a affects width rather than height.

4.4.2. a as the stretch factor

Five students described a as the "stretch factor." Four of the 5 students described the stretch factor as making the parabola wider or skinnier in a manner consistent with the student conceptions described above. One student, Mandy, also described the stretch factor as making the parabola taller, or pulling up the y-values:

Mandy: You take the original equation, which is x squared, and then if it's pulled by 1.5 [a new hypothetical value of a]. . .the original equation would be x squared, and when x is 10, y would be 100. If you multiply 100 by 1.5, you get 150.

Later, Mandy elaborated more generally on the influence of a, noting that the stretch factor affects the graph in the following way:

Mandy: Pull it [the graph] upward along the y-axis. . .it moves all the points, like the original points of the original equation being x squared, it moves those points up, higher, 1.5 times higher on the y-axis.

None of the students indicated that the stretch factor influenced the vertex of the parabola.

The students' reference to a as the stretch factor was expected given the strong emphasis on this language in the classroom discourse. In a prior DAA chapter on translations, stretches, and shrinks, the coefficient a was introduced as a vertical stretch or shrink. Following the text's lead, Ms. R initially referred to a as the "vertical stretch"; this occurred four times, once on the third day of instruction and three times on the fourth day. However, in the "Quadratic and Other Polynomial Functions" chapter, the text referred to a as the scale factor. The students conflated these terms by describing a as the "stretch factor", and the fourth time that the parameter a came up in discussion on the fourth day, the cooperating teacher adopted this term. She then continued to use "stretch factor" exclusively through the end of the unit, describing a in this manner 19 times when lecturing to the entire class. The students also described a as the stretch factor whenever its role arose in the classroom discourse. What was more surprising was that only Mandy referred to the stretch factor in terms of pulling the y-values up vertically, or more generally as influencing the graph vertically rather than in terms of width. Perhaps unintentionally, the term "stretch factor" ultimately came to refer to the width of the parabola through focusing phenomena such as the teacher and the students' gestures, the language of "tall and skinny" or "short and wide", and the ubiquitous use of the calculator to graph and view different parabolas.

4.4.3. a decides whether the parabola faces up or down

Five students also described a as determining whether the parabola will face "up" or "down", i.e., whether the parabola's vertex is a minimum or a maximum. For instance, Jack explained, "The negative flips it. Because otherwise you'd have a parabola like this [draws an upwards facing parabola] and then. . . the equation you put negative, then. . .it's like that [draws a downwards facing parabola]." Alexis described the role of a in terms of a shaping the parabola to resemble smile or a frown: "If you make it a negative number, the number will go, it'll be like a frown shape."

Alexis's description of a frown shape mirrored the teacher's language in the classroom:

Ms. R: Is it a smile or a frown?
Students: Smile.
Ms. R: How do you know it's a smile?
Amata: It's positive.
Ms. R: What's positive?
Amata: x squared.
Ms. R: The lead coefficient. The coefficient on the x squared.

The classroom dialog excerpt above showed the teacher focusing students' attention on a determining whether the parabola opens up or down. Thus it is not surprising that the students demonstrated these generalizations about the role of a in the interviews. However, given the classroom focus on the role of a as the stretch factor, influencing both the shape and direction of the parabola, the final generalization the students demonstrated about the role of a was particularly surprising. Specifically, 6 out of the 8 interview participants indicated that a was the "slope" of the graph:

4.4.4. a is the slope

Kordell used the calculator's "quadratic regression" program to write the equation $y = 1.5x^2$ to describe a table of values. The interviewer asked about the meaning of 1.5:

Int: What's the 1.5 have to do with anything in the table? Do you see the 1.5 anywhere?
Kordell: It's the slope.
Int: It's the slope?
Kordell: I think it's the slope. Yeah.
Int: Okay. So, what's the slope mean?
Kordell: The steepness of the graph. Like, what it goes up by.
Int: What it goes up by? Okay, so in this case, the graph goes up by 1.5?
Kordell: Yeah.
Int: So how does that work, like, can you draw a picture of what the graph looks like generally, what you found on your calculator?
Kordell: [Draws the graph shown in Fig. 6]
Int: Okay. So you've got that curve there. So. . .where's the 1.5 in terms of the slope on that kind of curve?
Kordell: I guess it'd be. . .I don't know. It's just the rise over the run. Slope.

Kordell's response may seem out of the ordinary, but it actually represents the views of the majority of the interview participants. Amata described a as "the constant rate", and explained "I think a affects how much it goes up by." When asked about a, Alexis responded, "it has to do with the slope" and Tashi replied, "how fast it rises."

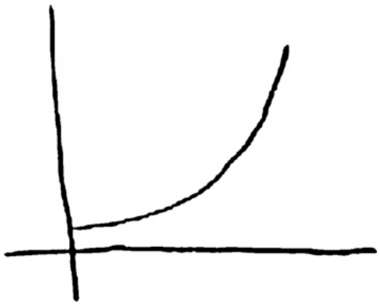


Fig. 6. Kordell's graph of a parabola.

Alexis not only stated that *a* was related to slope, but she also had a method for finding *a*. When given the rabbit pen task (Fig. 1), Alexis tried to write an equation to describe the parabola. She initially wrote $y = -ax^2 + 900$. Alexis explained that she added 900 "because it goes up 900", and she knew that *a* would be negative "because it's [the graph] inverted [upside down]." When asked how she could determine the value of *a*, Alexis responded, "You could do this over. . .rise over run." Alexis picked the two points (10, 500) and (20, 800). She then counted the "rise", i.e., the vertical distance between $y = 500$ and $y = 800$, and got 3 squares (appearing to ignore the scale of the *y*-axis). Alexis then counted the "run", i.e., the horizontal distance between $x = 10$ and $x = 20$, and got 1 square, again ignoring the scale. She concluded:

Alexis: So it's 3 over 1, which is basically 3.
Int: Oh, okay. How'd you get that 3 for *a*?
Alexis: I counted the number of squares it took for it to reach, like, the next block.

. . .

Int: Ah, so you're doing a rise over run kind of thing.
Alexis: Uh huh.
Int: And, which two points did you use?
Alexis: 10 by 500 and 20 by 800.
Int: Okay. So, I see you have 3 up like this and 1 over like that. Now, did you have to use these two points, or could you have used two other points?
Alexis: You could have used two other points.
Int: What would be two other points that you could use?
Alexis: Well, it could have been these two [refers to (0, 0) and (10, 500)]. . .
Int: So, if you used two other points, you're going to get some sort of different result?
Alexis: Uh huh. Like, if you used this point and this point [refers to (50, 500) and (60, 0)] you'll get a totally different answer.
Int: Ah, so how do you know which points to pick so that you'll get the right answer for *a*?
Alexis: Guess and check.

Alexis indicated that she could try several different values for *a*, plug each equation into her calculator, graph it, and then ultimately pick the equation that produced a graph most similar to the one on her paper. As we will describe in Part II, Alexis and the other students were taught to use this method for finding the equation for a linear function.

Although we were able to identify direct connections between the majority of the students' conceptions played by the parameters *a*, *b*, and *c* and what occurred in the instructional environment, the connection to slope was unexpected—parabolas do not have constant slopes, and the teacher, of course, did not describe the role of *a* or *b* as the slope of the parabola. Part II presents the results of our analysis in trying to determine the mechanisms behind the students' generalizations that the parameters *a* and *b* represented the slope of the graph.

4.5. Part II: how three focusing phenomena directed attention towards slope-like properties of parabolas

Although the students' conception of *a* as being related to slope was a surprising one, it was not a connection that we could discount because it occurred for a majority of the interview sample. Moreover, the students comprising the sample were the higher performing students in the class, identified by their teacher as those who demonstrated strong grades, regular classroom attendance, and good classroom participation. Our initial hypothesis to explain the students' slope connections was that they transferred between their understanding of linear functions and quadratic functions—this type of generalization from linearity has also been reported in other studies (Chazan, 2006; Buck, 1995; Hershkowitz & Schwarz, 1997; Zaslavsky, 1997). In this case, the interview students may have seen similarities and connections between the quadratic functions unit and their linear functions unit. Even though the connections were mathematically incorrect, they may have been valid connections from the students' point of view. Under the actor-oriented framework, transfer is sought by finding evidence of prior learning on current learning, regardless of whether that influence is viewed as mathematically valid from the researcher's perspective. Could it simply be the case that the students generalized a (and in the case of two students, b) as the slope because they were demonstrating transfer from linear functions?

The DAA text presented the equation of a line as $y = a + bx$, where a is the y-intercept and b is the slope of the line. Although we did not observe the classroom during the linear functions unit, the teacher indicated that she and the students conformed to this choice of symbolism. Identifying and focusing on b as the slope in $y = a + bx$ could partially explain the two students who believed that b was the slope in quadratic functions. Moreover, it is possible that the existence of the terms a and b in both the linear and quadratic equations could have contributed to the students' belief that an equation with an a and a b in it, such as $y = ax^2 + bx + c$, should have a slope.

Under this initial hypothesis, we turned to the classroom data to search for indications suggesting that the students might have demonstrated evidence that they were making connections between their current quadratic unit and their prior linear functions unit. What we found instead, however, were indications that features of the current instructional environment supported a focus on slope, specifically related to the parameter a. These were features that we did not set out to identify, so we then re-coded the data in order to examine the ways in which these features could have contributed to the students' beliefs that a was related to slope. Through this analysis, we identified 3 focusing phenomena: the use of linear analogies, the rise over run method, and the dynamic view. Below we present a representative example of each of these focusing phenomena. We will show that each focusing phenomenon encouraged an emphasis on slope-like features in relationship to the parameter a, which likely contributed to the students' generalizations that the parameter had a connection to slope.

4.5.1. Focusing phenomena #1: use of linear analogies

The DAA chapter on Quadratic and Other Polynomial Functions introduced the idea of quadratic functions through the finite differences method. Students encountered well-ordered tables in which x-values increased by a uniform value, and were instructed to calculate the differences between corresponding y-values. If the first differences were the same, the function was linear: "In modeling linear functions, you have already discovered that for x-values that are uniformly spaced, the differences between the corresponding y-values must be the same." (pp. 360–361). Students read that when second differences, or the differences of the differences, were the same, the function would be quadratic. This method for determining whether a table of values represented a quadratic function is similar to methods for calculating the slope of a linear function in a well-ordered table, and the cooperating teacher may have emphasized a conflation of the two methods.

Specifically, in an attempt to ground an understanding of the role of a in students' prior knowledge, the teacher relied on linear analogies when discussing features of quadratic functions. For instance, when explaining the role that a played in all three forms, the teacher made a comparison to rates of change for linear functions by describing a as a changing rate of change in contrast to a constant rate of change, or slope. In the following excerpt, Ms. R referred to the board where three graphs ($y = x^2$, $y = (x - 3)^2 - 2$, and $y = 2(x - 3)^2$) had already been drawn:

Ms. R: You know how you can use slope to graph a line? This [points to first graph] doesn't have constant slope, does it? But it's got a pattern that you would recognize from the constant differences that we did yesterday. So if we go over 1, we go up 1. [Starting at the vertex (0, 0), she moves her finger 1 unit to the right and 1 unit up to the next point on the graph, (1, 1)]. Now we go over 1 and we go up. . .[Starting from the point (1, 1), she moves her finger 1 unit to the right and 3 units up to the next point on the graph].
Students: 3.
Ms. R.: Then we go over one and we go up. . .
Students: [Calling out] 3. 9. 5.
Ms. R: 1, 3, 5, what would be next?
Students: 7.
Ms. R: 7, then 9.
Jack: 84.
Ms. R: Okay, how about here? [Points to the second graph]. We go over 1 and up. . .
Students: 1.
Ms. R: 1. Over 1 and up?
Students: Dos. [Spanish for 2].
Ms. R: 3. Over 1 and up?
Students: 5.
Ms R.: 5, same pattern [as in the first graph]. But on this last one [points to the third graph] we went over 1 and up. . .
Students: 2.
Ms. R: 2. Then we went over 1 and up. . .
Students: 6.
Ms. R: 6. Then we went over 1 and up 10. So what's happening to our finite differences?
Karsten: 'Cause you multiplied it by 2 so it [the first difference] goes up by 4 instead of 2.
Ms. R: Yeah. So you can see in the equation now, that 2 is what's causing that vertical stretch.

By pointing to the different y-value increases on the graph when moving from one point to the next, the teacher appeared to intend to draw the students' attention to the fact that the first differences for parabolas exhibit a patterned behavior similar to those of linear equations and that the parameter a and the slope, respectively, determine these features. However, there is some evidence that this contributed to the students' conflation of the two ideas, as shown in Amata's description of the parts of a quadratic graph:

Int: Okay. So I'd like you to draw a quadratic function in this box and just make it nice and big so that we can see really well.
Amata: [Draws a parabola similar to $y = -x^2 + 10$].
Int: Okay. So what makes this quadratic?
Amata: Um. . .it's a parabola, so it has the x^2 [points to general form of a quadratic, written earlier], it's got c [points to y-intercept], it's got a slope [hovers hand over paper], and it has a stretch [places two fingers on graph at each root].
Int: Hmm.
Amata: And, it has all the components it needs.
Int: Well, that's a lot of components! Can you point out all the important parts of this graph?
Amata: The roots [circles the x-intercepts], the vertex [circles the vertex].
Int: Uh huh.
Amata: And, somehow, we magically circle the slope [circles a part of the parabola near $(-6, 6)$].
Int: What's this part here?
Amata: I just randomly circled some place that looked like it was the slope.
Int: Okay, so, um, where's this like, how's that the slope, exactly?
Amata: Um. . .the slope because it's the certain amount of squares [draws a grid within the circle] that goes, that goes down for it to reach one [draws a dot at one intersection of the parabola with the grid] to the other [draws another dot at the intersection of the parabola with the grid].

In referring the students to the differences method in relationship to a, which was geared only towards determining the degree of a polynomial in DAA, the teacher may have strengthened a connection between the meaning of a and previous methods for finding slope values.

Further evidence that both the students and the teacher may have connected a to slope occurred on a day when the class discussed the equation $y = -2(x - 5)^2 + 8$. The students had identified -2 as the stretch factor, and the teacher asked what that meant:

Ms. R: But what does that mean?
Jason: It goes up or down.
Ms. R: What do you mean when you say it goes down?
Jason: Like. . .down 2, over 2. . .
Ms. R: Like 2 would be slope? Okay.

Finally, one major emphasis in the quadratic functions unit was the symbolic manipulation skills needed to transition between the general form, vertex form, and factored form of an equation. Because the students' algebraic manipulation skills were fairly weak, the teacher relied on their greater comfort and familiarity with manipulating linear functions in order to support their abilities to manipulate quadratic functions:

Ms. R: Figure out what these, the general form of these [equations] will be. This is something called vertex form. Do you all remember what general form looks like for parabolas?
Students: Quadratic?
Ms. R: Yes. *ax* squared plus *bx* plus *c*.

[Students struggled on their own to convert equations in vertex form to general form].

Ms. R: [Writes a linear equation, $2x + 5y = 10$, on the board]. What would you do if you want to put that in y equals form? Slope intercept form.
Student: Minus $2x$ and divide by 5. [Writes $y = (-2/5)x + 2$ on the board].
Ms. R: Okay. Are these the same line?
Students: Yeah.
Ms. R: Yeah. Are they the same form of the equation? No. So you can write this [referring to a quadratic equation in vertex form] in y equals *ax* squared plus *bx* plus *c* form.

Through an emphasis on the comparisons between the linear form of an equation and the quadratic general form, the students may have connected the first term of the general form, a, with the first term of a standard linear form of an equation, which is its slope. In general, the use of linear analogies to aid in students' understanding of quadratic functions and the role

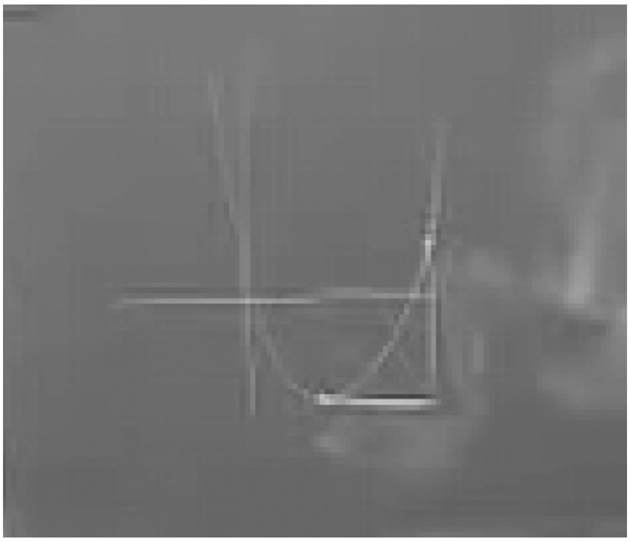


Fig. 7. Classroom drawing of a parabola.

of a could have contributed to a conflation between a property found in linear functions, constant slope, and the property described by a in the general form of a quadratic equation.

4.5.2. Focusing phenomena #2: rise over run method for finding a

On the fifth day of the unit the participating teacher introduced a method for finding the value of a given a graph of a parabola:

Ms. R: I wanted to show you that there is one other way to figure out what a is. If you have some points on the graph—[points to a graph on the board, see Fig. 7]. If you have some points on the graph, what's important about this point right here? [Points to the vertex].
Students: It's the vertex.
Ms. R: It's the vertex. And then this is just some random point [points to a spot on the right side of the parabola above the vertex]. Can you make up numbers, nice numbers, that might match this point [gestures towards the vertex] right here?
Student: Negative. . .wait. (3, −3).
Ms. R: Okay, let's call this (3, −5). And if that's (3, −5), what does this look like it might be over here? [Gestures to the second point she had indicated].
Mandy: 7. . .
Ms. R: 7. Okay, 7 what?
Mandy: 2.
Ms. R: 2. All right. So, here's my question. What's this distance if we're going from 3 to 7?
Students: 4.
Ms. R: 4. And what's this distance, if we're going from −5 to 2?
Students: 7.
Ms. R: 7. Here is your vertical stretch factor a. We take the height, we divide it by the width squared. [Writes "$a = 7/16$"]. So it would be 7 sixteenths. It's not very nice, and what I don't like about it is that you don't just look at this and sort of say, well, of course that's what you do.

The participating teacher continued to emphasize this method for finding a, but the students never inquired as to its origin or why it made sense. Moreover, the focus in the above episode was similar to each case in which the teacher used the graphical method for finding a: she paid a great deal of attention to locating two nice points on the graph, and drew in the "height" and the "width" marks from one point to another. This focused attention on the value of the vertical and horizontal distances, and the teacher paid relatively little attention to the final step of the process, in which one would calculate a by dividing the "height" by the "width" *squared*. Given its similarity to finding the value of a slope and the focus on calculating distances, some students may have begun to associate the a value with the "slope" of the parabola.

The DAA text also presented the graphical method for determining the value of a (Fig. 8). Notice that again the "rise" of 6 and the "run" of 3 are drawn into the second graph, which is presented as a transformation of the parent graph $y = x^2$. The text included the equation $y = 6/9(x - 4)^2 - 2$, or $y = 2/3(x - 4)^2 - 2$ and explained "Notice that the horizontal and vertical scale factors are now represented by one vertical scale factor of 2/3. This coefficient, a/b^2, combines the horizontal and vertical scale factors into one vertical scale factor, which you can think of as a single coefficient, say a." It may be possible

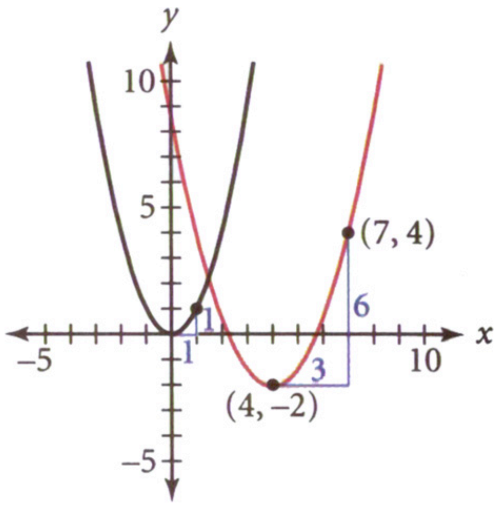


Fig. 8. Illustration of the graphical method for determining the value of a.

that the visual representation of the vertical and horizontal distance between the two points was more salient than the description of the coefficient as a/b^2, which was buried in the text. Moreover, the emphasis on calculating the vertical and horizontal distances is visually similar to the way in which DAA presented the method for finding the slope from a linear graph (Fig. 9).

Recall Alexis's method for finding the value of a from a graph. It mirrored almost exactly Ms. R's method as demonstrated in class, but when she calculated a, she divided the rise by the run instead of by the square of the run. The similarity between the graphical method for finding the value of a and the method for finding the slope of a line could have contributed to Alexis's belief that a "has to do with the slope."

4.5.3. *Focusing phenomena #3: viewing a as dynamic rather than static*

Classroom discussions about the role of a as affecting the shape of the graph most often focused on a as determining whether a parabola would be tall and skinny or short and wide. However, the teacher and students' language and gestures emphasized a dynamic interpretation of a, in which it influenced either the steepness of the graph, or how fast the graph would rise. For instance, in the episode above when Amata asked what the value of a would do to a parabola, the teacher's moving gestures emphasized the role of a as dynamically increasing the steepness of a parabola. She completed her explanation by stating, "It changes how fast it goes up."

This dynamic language was also reflected in the students' interview language, for instance, when Alexis explained that she chose 1 for a "because its slope is going up." When asked how the graph would change if a were changed to 10, Alexis explained, "It would be really skinny and the slope would be really steep", suggesting that she viewed the skinny/wide distinction of a parabola as connected to its steepness, or its slope. Alexis also visually indicated that she saw the slope as the steep part of the graph (see Fig. 10): "I just randomly circled some place that looked like it was the slope."

Similarly, when the interviewer asked Andrew whether a affected the graph, she explained "Yeah, how fast it goes up." When Amata was asked what the a did, she said, "I think the a is the rate which it goes up", and Tashi explained that a higher value for a affects how the parabola goes up, "And if it's like a bigger number, it just like, you know 1 or a prime number. . .real

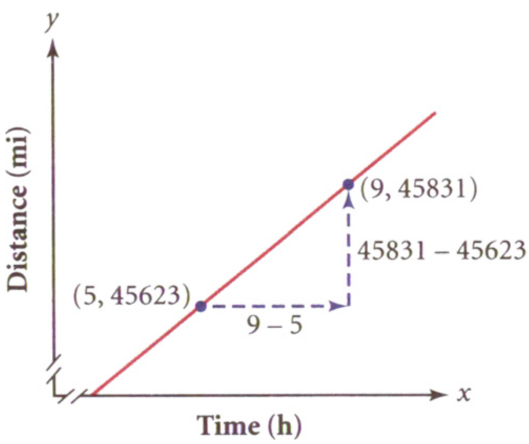


Fig. 9. Illustration of finding the slope from a graph.

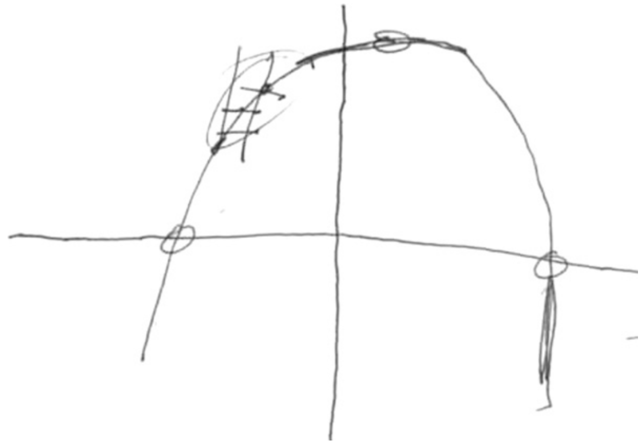


Fig. 10. Alexis's circled area of the parabola that indicates its slope.

number, that, a real number, um, it closes up more but it goes up faster, so basically it just goes up faster." He elaborated that a very large number for a, such as 5, would make the parabola "just go up really really fast."

In combination with the teacher's language, the students' language in the interviews and in the classroom revealed a focus on the value of a as influencing the graph of a parabola in a dynamic manner. A large value for a will make the parabola "go up" faster or at a higher rate. This emphasis on a dynamic role for a could have contributed to a conflation of the role of a and the role that slope plays. Both appeared to have been referenced in terms of how fast the graph "goes up." Although the DAA text relied on the term "rate" when describing slope, the teacher indicated that she and the students made use of the less formal "goes up" language when referring to slope, which was consistent with the students' language when discussing linear functions and slope in the interviews. For instance, Corine and Kordell both referred to a slope value of 0.25 in a linear function as meaning that "it goes up by 0.25 each time." Although we did not videotape the linear functions unit, there were enough instances in the interviews and informal classroom conversations in which the students referred to slope values as how fast the line or function "goes up" to suggest that this language may have been connected to the idea of slope for the students. Other studies (Lobato et al., 2003) have similarly identified the connection between slope and "goes up by" language. Therefore, using the same language to refer to both slope and the role played by a could have contributed to a generalization that a was the slope of the parabola.

5. Discussion

Our initial hypothesis to explain students' generalizations that the parameters a or b represented the slope of a parabola was that students simply generalized from their experiences with linear functions. Researchers have found that students will give a linear function as a prototypical example of a function (Schwarz & Hershkowitz, 1999), draw analogies between linear and non-linear functions (Buck, 1995; Chazan, 2006) and inappropriately draw on their linear-function understanding to try to solve tasks for non-linear functions (Hershkowitz & Schwarz, 1997; Zaslavsky, 1997). One interpretation of the students' behavior and language in the classroom discussions and the interviews is that they too relied on an idea of linearity as a prototypical function, generalizing from their understanding of linear functions to make sense of the parameters in $y = ax^2 + bx + c$.

However, our analysis suggests that the focusing phenomena present in the classroom environment served to support and reinforce students' tendencies to generalize from linearity. The teacher's and the text's reliance on connections to linear functions was intended to not only build bridges between an understanding of linear and quadratic functions, but also serve as a way to contrast features that differed across the two function families. However, the students' reactions to these analogies and contrasting cases indicated a conflation of linear and quadratic properties, rather than a distinction between them. Moreover, the use of the rise over run method for finding a and the dynamic language of the role played by a served to further reinforce connections between the meaning of a and the notion of slope for linear functions. Ultimately, these focusing phenomena competed successfully with the more direct focus that the teacher intentionally emphasized for a as indicating the vertical stretch and the orientation of the parabola.

There is some literature support for the strategy of building upon students' existing knowledge of linear functions to develop an understanding of quadratic functions (Buck, 1995; Movshovitz-Hadar, 1993). One can, for instance, introduce the idea of a quadratic function as the product of two linear functions, leading Movshovitz-Hadar to suggest that "it is unnecessary to keep in mutual isolation the study of linear and quadratic functions." (p. 292). The results of this study lend credence to that notion: if students are already likely to generalize from their understanding of linear functions, then it is important recognize this tendency and find ways to allow it to be a productive support for quadratic understanding rather than a source of impedance. At the same time, the manner in which teachers attempt to build on an understanding of linear functions could also inadvertently support incorrect generalizations. For this reason, we must consider the potential ramifications of any instructional focus, either explicit or implicit.

5.1. Shifting the focus from representations to relationships

One way to better help students understand the connections between algebraic and graphical representations of quadratic functions would be to change the object of focus and the accompanying focusing phenomena supporting generalizations about these connections. In Ms. R's classroom, the focus was on the symbolic representation $y = ax^2 + bx + c$ and the ways in which various forms of this representation affected the accompanying graph; the symbolic representation itself was consistently emphasized, rather than any phenomenon described by that representation. Although the students did engage in investigations such as rolling objects down inclined planes, these investigations occurred *after* the general form $y = ax^2 + bx + c$ had been introduced and solidified. The relationship between distance and time then became an example of quadratic data, rather than a phenomenon that students could mathematize and explore in order to develop an understanding of the nature of quadratic relationships. Moreover, since any variants such as $y = ax^2$ or $y = ax^2 + c$ were presented after the general form as special cases, the students experienced difficulty teasing apart the effect of each parameter, much less how they interacted with one another.

Because the classroom focus remained on the various symbolic forms of quadratic functions, the students did not have any opportunities to investigate the nature of quadratic growth or explore the ways in which different types of quadratic relationships might be represented, both algebraically and graphically. Given the lack of these opportunities, it is not surprising that the students had to rely on textual and teacher-led cues to determine the effects of changing the parameters a, b, and c. In contrast, we propose a shift in focus away from the algebraic and graphical representations themselves and towards quantitative relationships (Thompson, 1994) that are quadratic in nature.

Imagine a classroom focus on the relationship between quantities, and an environment that encourages exploration of these relationships. One example of an accessible quadratic relationship is the relationship between the length and the area of a square (or a rectangle conforming to a set of restrictions). For instance, students could begin by examining the relationship between the length and the area of a square in order to investigate the nature of quadratic growth for the function $y = x^2$. Students could generate their own algebraic and graphical representations of this relationship as they examine relationships between quantities. Students could then move to considering relationships between the length and area of other shapes, for instance, a family of rectangles in which the length is twice the width ($y = 2x^2$), or a family of rectangles in which the length is always 5 units greater than the width ($y = x^2 + 5x$), or a family of shapes that constitute a square of any size plus 5 individual units ($y = x^2 + 5$). In this manner, students could investigate the role played by a in terms of its influence on the quadratic relationship in question: the a in $y = ax^2$ would represent how many times longer a rectangle is than its width. In addition, the general form of a quadratic function could be built up in pieces by first examining $y = ax^2$, then investigating $y = ax^2 + c$, before finally moving to $y = ax^2 + bx + c$, an approach advocated elsewhere in the literature (Movshovitz-Hadar, 1993).

Once the focus is shifted towards relationships between length and area, teachers could include comparisons to linear data by introducing an investigation of the relationship between length and perimeter. When examining how perimeter changes, students can observe that when the length of a square (for instance) doubles, so does the perimeter, but the area increases by the square of the original side length increase. Students could also explore the constant increase in perimeter when units are added to the side length, and contrast this with the non-constant increase for area. An explicit focus on the constant nature of linear growth versus the changing nature of quadratic growth could help students build on their understanding of linear functions in productive ways, rather than encouraging inappropriate generalizations based on surface-feature similarities between algebraic representations or algorithms for determining the values of parameters.

We agree that instruction of quadratic functions should not ignore students' linear function understanding, but should instead build on students' existing ways of knowing and capitalize on students' tendency to generalize from linear functions. However, the results of our study showed that the manner in which one attempts to build on linear function knowledge can produce unintended consequences. In Ms. R's classroom, the implicit focus on the slope-like properties of the parameter a encouraged a conflation between the role of a and the idea of slope as a constant increase. This does not imply, however, that any connection to linear functions will encourage inappropriate generalizations. Instead, we suggest a focus on the quantitative relationships that can be represented by both linear functions and quadratic functions, and an examination of the ways in which quadratic growth differs from linear growth. Given a focus on relationships between quantities, students will be poised to make sense of various algebraic and graphical representations in relationship to the phenomena they represent, and will be better prepared to examine functional relationships that involve non-constant change.

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