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How "Focusing Phenomena" in the Instructional Environment Support Individual Students' Generalizations

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This article sets forth a way of connecting the classroom instructional environment with individual students' generalizations. To do so, we advance the notion of *focusing phenomena,* that is, regularities in the ways in which teachers, students, artifacts, and curricular materials act together to direct attention toward certain mathematical properties over others. The construct of focusing phenomena emerged from an empirical study conducted during a 5-week unit on slope and linear functions in a high school classroom using a reform curriculum. Qualitative evidence from interviews with 7 students revealed that students interpreted the m value in $y = \mathrm{b} + \mathrm{m}x$ as a difference rather than a ratio as a result of counterproductive generalization afforded by focusing phenomena. Classroom analysis revealed 4 focusing phenomena, which regularly directed students' attention to various sets of differences rather than to the coordination of quantities.

There is a growing concern among mathematics educators that many students leave school unable to connect school mathematics to work or everyday settings. As a result, many of the major curriculum development projects in the United States have made connections with real-world situations a fundamental feature of their materials. This article examines how students in one of the reform mathematics programs, the Core-Plus Mathematics Project (CPMP, Coxford et al., 1998), generalized their understandings from realistic situations to novel settings and how the classroom environment afforded those generalizations.

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We chose to investigate the mathematical topic of slope as the rate of change of a linear function because it is conceptually complex for students, rich in terms of real-world connections, and featured as an important topic in both CPMP and the *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989).[1] This study is not an evaluation of CPMP. We selected CPMP because it showed promise for supporting the development of productive generalizations related to slope and linear functions. Specifically, the curricular materials explore functions as a way to describe dependency relations in complex, real-world situations and treat slope as a rate of change of covarying quantities in multiple, real-world settings. We expected this approach to increase the likelihood that students would be able to negotiate the quantitative complexity of unfamiliar situations. It is a vast improvement over the ubiquitous treatment of slope as a counting technique used to determine the steepness of a line in a coordinate grid system, which can make applications not involving graphs or steepness difficult for students.

The purpose at the outset of the study was to examine the nature of the generalizations students formed about slope and linear functions as a result of their interactions with a reform curriculum that regularly develops concepts in real-world settings. Halfway through our investigation, we were surprised to find that all of the students in the interview sample appeared to interpret m in $y = \mathrm{b} + \mathrm{m}x$ as a difference rather than a ratio and to generalize that understanding to novel settings. In response, we conducted a detailed analysis of the classroom videotapes to understand how the instructional environment supported these unintended and unwanted generalizations. We thought that by understanding this connection we would gain insight into how to design an instructional intervention. A powerful construct, which we call *focusing phenomena,* emerged from our analysis and allowed us to connect students' generalizations with the classroom instructional environment. Our goals in this article are to (a) introduce the notion of focusing phenomena, (b) report the findings from our analysis of the interview participants' generalizations, (c) identify four focusing phenomena from our analysis of the classroom data, and (d) articulate a set of connections between the focusing phenomena and students' generalizations.

CHARACTERIZATION OF FOCUSING PHENOMENA

Focusing phenomena are observable features of the classroom environment that regularly direct attention to certain mathematical properties or patterns. Focusing

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[1]The National Council of Teachers of Mathematics (1989) *Curriculum and Evaluation Standards for School Mathematics* is referenced rather than the more recent *Principles and Standards for School Mathematics* (NCTM, 2000) because the Core-Plus Mathematics Project was developed to meet the 1989 standards and was written prior to the development of the 2000 principles and standards.

phenomena emerge not only through the instructor's behavior but also through co-constructed mathematical language, features of the curricular materials, and the use of artifacts such as graphing calculators.

The construct of focusing phenomena is rooted in a situated view of the abstracting process. Abstracting involves the identification of regularities in one's activities, the isolation of certain properties, and the suppression of other details (Frorer, Hazzan, & Manes, 1997; Harel & Tall, 1991). For example, a young child constructs "eightness" by identifying a property common to her actions on sets of eight cups, eight bowls, and eight bottles (namely, cardinality), while ignoring other properties (e.g., the property of holding liquid) and then considering the common feature in isolation. There are several common interpretations of the meaning of abstraction. For the purposes of this study, we are interested in the process of reflective abstraction, which involves focusing on and isolating items (such as mathematical properties or regularities) from the experiential flow, re-presenting those aspects of experience, and coordinating them into new experiences (von Glasersfeld, 1995). We are not using abstraction as it is commonly used to suggest a quality of an object or system as being nonconcrete, decontextualized, or theoretical.

We also follow Hershkowitz, Schwarz, and Dreyfus (2001) in their efforts to situate the abstraction process. Specifically, they point to the importance of attending to the multifaceted context in which abstracting occurs:

> A process of abstraction is influenced by the task(s) on which students work; it may capitalize on tools and other artifacts; it depends on the personal histories of students and teachers; and it takes place in a particular social and physical setting. (p. 196)

The notion of focusing phenomena accounts for the multiple agents that are involved in the activity of directing students' attention to particular aspects of mathematical activity.

Our examination of the links between focusing phenomena and individual students' generalizations is founded on the theoretical connection between the processes of abstracting and generalizing. Abstracting is one of the processes involved in the creation of a mental structure (von Glasersfeld, 1995), and generalizing involves the extension of an existing mental structure to new objects and situations (Harel & Tall, 1991). Consider the well-documented example of children's extension of the concept *multiplication makes bigger* from the domain of whole numbers to rational numbers (Greer, 1992). Years of experience with whole numbers likely directs students' attention to the pattern that the product is always larger than either factor when both factors are whole numbers. Therefore, students generalize on the basis of that experience. The curricular attention on whole numbers for years prior to the introduction of rational number multiplication can be thought of as a focusing phenomenon. This article investigates the many other ways in which the classroom instructional environment, especially the teacher's actions, supports particular generalizations by directing students' attention toward specific mathematical regularities and properties over others.

We conclude this section with a note about our use of terminology. Multiplication makes bigger is frequently and understandably referred to as an *overgeneralization* because mathematically incorrect statements often result and because it is a generalization to a larger domain that subsumes the previous domain. However, the term overgeneralization may direct more attention to what is incorrect than to what is generalized. Smith, diSessa, and Roschelle (1993) similarly argued that researchers often focus on the "mis" part, rather than on the "conception" part, of misconception. In this article, we report the generalizations that students develop, even when those ideas are mathematically limited or incorrect in a larger context. If we excluded mathematically incorrect generalizations, then we would limit our ability to examine the links between the focusing phenomena in the classroom instructional environment and students' generalizations.

CONCEPTUAL ANALYSIS OF SLOPE

Much of our analysis involves making judgments about students' conceptual understanding of slope. Accordingly, we briefly describe our conceptual orientation toward slope for the purposes of this study. One concept fundamental to the understanding of slope is the creation of a ratio of the variation of one quantity to the associated variation of another quantity, where the two quantities covary. Thompson (1995) warned of the dangers of confusing mental operations with arithmetic operations. Simply calculating a quotient in the slope formula does not insure that students have mentally formed slope as a ratio. One way to create a ratio is through the mental operation of multiplicative comparison, namely, comparing how many times greater is one quantity than another (Thompson, 1994). Another way to create a ratio is by forming a "complex unit built of two composite units" (Lamon, 1995, p. 171). For example, a speed situation may give rise to the ratio "10 cm traveled in 4 sec." Ten centimeters is a composite unit: Ten centimeters are considered as 1 ten-unit. Likewise, 4 sec is a composite unit. The ratio 10:4 is a complex composite unit formed by relating the composite units 10 and 4. The ratio 10:4 can then be operated on as a single entity, for example, forming the ratio 20:8 as two of the unit 10:4.

Quantitative reasoning is important in this study in part because of the emphasis on real-world functions in CPMP. According to Thompson (1994), quantities are constituted in people's conceptions about measurable attributes of objects, events, or situations such as length, age, or time. Quantities are not numbers, although numbers can be values of quantities, such as 6 feet or 40 years. Several aspects of quantitative reasoning with slope are important for this study. First, although slope has its reference in the geometry of lines, it is also the ratio of the change in the dependent variable to the change in the independent variable and therefore has meaning in tables, physical situations, and verbal descriptions of functions. Second, in a realistic situation, slope involves the formation of a ratio as the measure of a given attribute (what Simon & Blume, 1994, call a "ratio-as-measure"). Researchers have investigated the process of creating ratios as measures of particular attributes such as constant speed or the steepness of a ramp (Lobato & Thanheiser, 2000, 2002; Simon & Blume, 1994). A third complexity of slope is illustrated by the difficulty children have associating the invariance of a ratio with the constancy of a quantity, such as taste or speed (Harel, Behr, Lesh, & Post, 1994). Relating the constancy of the attribute measured by slope to the invariance of slope as a ratio constitutes an important part of understanding slope in real-world settings. Because we are interested in how students develop a deep understanding of slope, we examine the multifaceted nature of operating with slope fluidly in quantitatively complex situations.

RESEARCH METHODS

Participants and Data Collection

This study was conducted as part of a continuing 5-year research project, the Generalization of Learning Mathematics Project. The collaborating school partner for the project is an urban high school in a southwestern city in the United States. Seventy-five percent of the school's 2,436 students are Hispanic, and the remaining students are Filipino (13.5%), White (5%), African American (3.5%), Asian or Pacific Islander (2.5%), and Native American (0.5%). Twenty-five percent of the students are classified as limited English proficient. The students perform below the nation's average on standardized achievement tests for mathematics and reading. The school was selected in part because it was one of only two high schools in the greater metropolitan area using the CPMP materials.

The collaborating teacher, whom we will call Ms. R, was recruited for the study in part because she met the criteria set by Huntley, Rasmussen, Villarubi, Sangton, and Fey (2000) for appropriate implementation of CPMP. That is, she followed the intended curriculum, used graphing calculators in the classroom, and encouraged cooperative learning strategies in heterogeneous groups. Ms. R had been recommended by local mathematics education leaders as someone who was interested in reform and who valued the development of conceptual understanding. She was enthusiastic about the CPMP program, particularly because it enabled her students to make connections between mathematics and their daily lives. Ms. R had taught for 6 years at the time of the study and acted in a respectful and caring manner toward her students. Data collection occurred in the Course 1 class (n = 36) that Ms. R deemed the most productive in terms of classroom management and students' readiness to learn.

Both individual interview data and videotaped classroom data were collected. Two sets of semistructured interviews (Bernard, 1988) were conducted. The first interview occurred approximately 4 weeks into the instructional unit, after the relevant topics of slope and the slope-intercept form of a linear equation had been developed. The second interview occurred 3 weeks after the completion of the instructional unit. Each interview lasted about 45 min and was videotaped. Great effort was extended to establish rapport with all students and particularly with the interview participants. Classroom observations occurred for 2 weeks prior to collecting videotaped data, so that the students were familiar with the researchers by the time the interviews occurred. As evidence of this familiarity, all of the interview participants freely responded to some interview questions by simply stating that they did not know the answer. Therefore the researchers were able to attribute a reasonable degree of validity to the responses that the students did provide, and made every effort to minimize responses based on social pressure.

Seven students participated in each interview. Two students from the first interview were unable to participate in the second interview and were replaced by two additional students. The interview sample was theoretically relevant rather than random. Because our intent was to examine the generalizations that students constructed in this instructional environment, we selected students who appeared to be capable of making sense of the content. Many of the students in the class had sufficiently poor mathematics skills that they appeared to be unprepared for the CPMP program. Therefore, all interview participants were identified by the classroom teacher as students who were prepared for CPMP, based on the criteria of high mathematics grades (in the A–C range), willingness to participate in classroom discussions, and discipline in completing homework assignments. The sample included three of the four top-performing students in the class. Gender-preserving pseudonyms have been used for all participants.

Videotaped classroom data were collected for 5 weeks by placing one camera in the back of the room. During whole-class discussions, the camera captured the activity of the teacher and the students who presented their work at the blackboard located in the front of the classroom. During small group work, the camera captured the activity of a target group of four students. The class met three times per week, once for 60 min and twice for 90 min. The target group's membership was fairly stable but not identical throughout the 5-week period. The target group members were identified by the teacher as students who were articulate, participated in class, and were earning a course grade of C or above. Because these same criteria were used to recruit interview participants, all four members of the target group were invited to participate in the interviews, but only one student agreed to do so. By not requiring that interviewees comprise the target small group, we were able to investigate the thinking of a greater number of students. Three researchers were present during each class session. One operated the video camera, another took field notes, and the third took field notes and visited the target small group at least once per class session. He or she tried to engage the students in mathematical discussions and probe their understanding without disrupting the normal flow of the class and without intentionally offering new information.

Interview Instruments

In each interview, students were asked a series of questions involving linear function concepts arising from a single situation: a boogie board rental situation for Interview 1 and a leaky faucet situation for Interview 2. Situations and questions were not designed to evaluate the CPMP curriculum and therefore did not span the range of tasks found in the CPMP text. Rather, each task was constructed to identify what the students generalized. We used the boogie board data from the CPMP textbook but presented them in a graph to see how students generalized their reasoning with a familiar data set from a tabular to a graphical setting (see Figure 1).

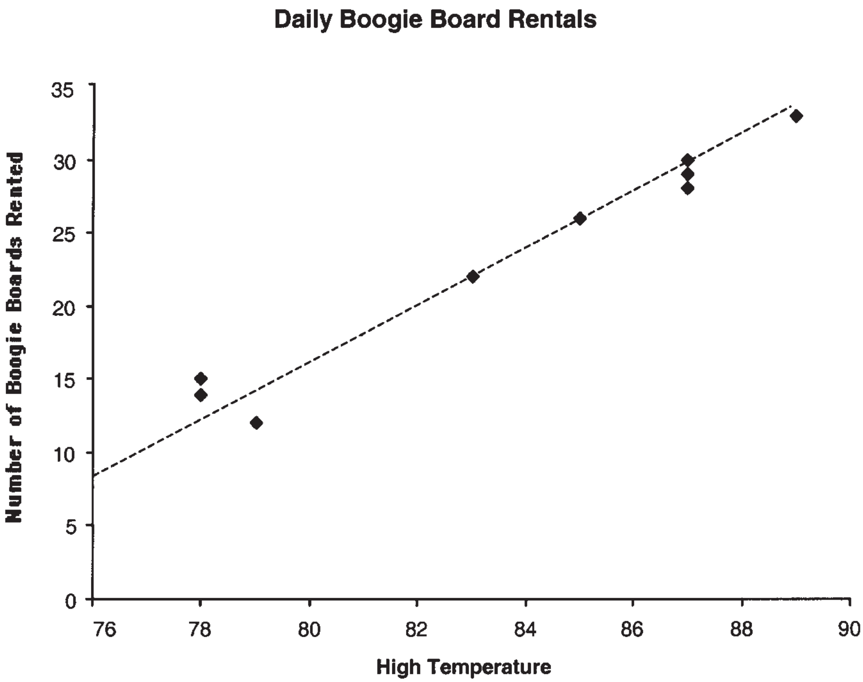

FIGURE 1 Situation presented in Interview 1 (with Carissa's line of best fit indicated by the dotted line): The *Oh-So-Cool Surf Company* rents boogie boards at Long Beach. Naturally, their business is affected by the weather. This graph shows boogie board rental data from nine weekend days during July and August 1999. Adapted from *Contemporary Mathematics in Context, Course 1 Part A* (p. 171), by A. F. Coxford, J. T. Fey, C. R. Hirsch, H. L. Schoen, G. Burrill, E. W. Hart, et al., 1998, Chicago, IL: Everyday Learning Corporation. Copyright 1998 by The McGraw-Hill Companies. Adapted with permission.

We designed the leaky faucet task to investigate what students generalized about rates of change from their work with well-ordered tables in CPMP to a tabular situation with non-uniform intervals (Figure 2). (By "well-ordered tables," we mean tables presenting the values for the independent variable in uniform intervals.) The leaky faucet situation was also different from any context the students had encountered in their textbook.

The analysis reported in this article is part of a larger study that examined students' generalizations related to important linear function concepts. This article presents the results of students' responses to a request to write and interpret an equation for the graph provided in Interview 1 and for the table presented in Interview 2.

Overview of the CPMP Instructional Unit

The classroom observations occurred during the first 5 weeks of instruction in Course 1, Unit 3, "Linear Models." We refer to the portion that we observed as the instructional unit although it was about two thirds of Unit 3; the remainder of Unit 3 addressed systems of linear equations. The unit consisted of several multiday lessons in which ideas were regularly developed through investigations of real-world situations. Figure 3 provides an overview of the mathematical ideas and the realistic situations as they were explored in the classroom. The instructional unit was organized around understanding linear equations in the slope-intercept form and making connections among realistic situations, tables, equations, and graphs.

Time	Amount of Water
7:00 a.m.	2 ounces
8:15 a.m.	12 ounces
9:45 a.m.	24 ounces
2:30 p.m.	62 ounces
5:15 p.m.	84 ounces
6:00 p.m.	90 ounces
9:30 p.m.	118 ounces

FIGURE 2 Situation presented in Interview 2: Cassandra decided to see how fast her bathtub faucet was leaking. She got a large container and put it under her faucet when she got up in the morning, and then checked periodically during the day to see how much water was in the container. She wrote the times and the amounts down. They are recorded in the table above.

	Situations	Topics	Day	Mathematical Ideas
Week 1	Overhead Projectors	Exploration of Linear Data	1	Collect data and make predictions
			2	Write equation and graph for data
	TV Ratings		3	Explore relationships between two quantities
Week 2	Concert Attendance		4	Predict using tables and scatter plots
			5	Find a linear model for a scatterplot
			6	Use the calculator to locate a line of best fit
Week 3	Rubber Bands and Springs	Exploration of Rate of Change	7	Introduce rate of change as $\frac{\Delta y}{\Delta x}$
			8	Calculate and explore rate of change
Week 4	Cost For Soda Machines	Slope, y-intercept and $y = \mathrm{b} + \mathrm{m}x$ Formally Defined	9	Define slope as a constant rate of change
			10	Write equations in $y = \mathrm{b} + \mathrm{m}x$ form.
	Interview 1 Conducted After Day 10 and Before Day 14			
	Equations in Graphical Settings		11	Compare $y = \mathrm{b} + \mathrm{m}x$ and now/next forms of equations
Week 5		Connections Among Tables, Equations and Graphs	12	Graph $y = \mathrm{b} + \mathrm{m}x$ equations
			13	Explore the effect of changing b and m on graphs of equations
	Speed With Motion Detectors		14	Connect speed phenomena with equations and graphs
	Interview 2 Conducted 3 Weeks After Day 14			

FIGURE 3 Overview of the development of linear equations and slope in the Core-Plus Mathematics Project unit.

The CPMP approach relies on modeling and linked representations to develop algebraic ideas. For example, the unit began with a 3-day activity in which students explored a linear relation between the distance of an overhead projector from a wall and the enlargement factor between the length of an image placed on the projector and the length of its projected image. By representing certain aspects of the problem situation with tables, students had the opportunity to experience an approximately linear relation between the distance of the projector from the wall and the enlargement factor. Students used graphing calculators to locate linear models for various scatterplots of realistic data, thus gaining informal experiences linking linear equations with graphs. Students also explored the rate of change of a variety of linear functions in real-world situations before the term *slope* was introduced and defined.

Lessons were designed to promote cooperative learning. We observed the use of small group activities during 12 of the 15 class sessions. The textbook included many exploratory and discovery-oriented problems. The teacher blended a variety of instructional formats—whole class discussion, data collection, small group activities, and some direct instruction. The teacher faithfully followed the CPMP curriculum, except for supplementing with some additional practice and with a lesson on motion detectors.

Data Analysis

We began by analyzing transcripts of the interview data. Our goal was to infer categories of meaning for the *m* value in the students' linear equations. Analysis of the interview data followed the interpretive techniques in which the categories of meaning were induced from the data (Glaser & Strauss, 1967; Miles & Huberman, 1994; Strauss & Corbin, 1990). Each of the authors developed analytic categories and coded the transcripts from Interview 1 separately. We sorted out troublesome cases through a process of argumentation and settled on a common set of categories. We tested for the validity of the categories by reviewing the transcripts for Interview 1, seeking confirming as well as disconfirming evidence. We then applied the categories to the data from Interview 2 and made appropriate modifications to the categories.

Once we identified categories of meaning for *m*, we analyzed the classroom videotapes. Our primary goal was to understand how the instructional environment may have supported the development of the students' meanings for *m*. We limited the scope of our investigation to the ways in which the instructional environment directed students' attention toward certain mathematical properties over others. The classroom analysis draws on the constant comparative method used in the development of grounded theory (Glaser & Strauss, 1967; Strauss & Corbin, 1990). It involves constantly comparing data as they are analyzed against conjectures generated thus far in the data analysis and revising conjectures accordingly (Cobb & Whitenack, 1996; McClain & Cobb, 2001). In addition to being compared against conjectures, incidents of the participants' activity in the classroom are also compared against one another. This gives rise to general themes or patterns and leads to an ongoing iterative refinement of the broad theoretical categories developed from the data (Cobb, Stephan, McClain, & Gravemeijer, 2001).

Next, we tried to explain how the instructional environment supported each of the categories of meaning for *m* by looking for regularities in the ways in which the environment focused students' attention on certain mathematical properties. The interview participants referred to the *m* value in their linear equations as "what it goes up by." Therefore, we began by initially coding the classroom transcripts for each instance of the "goes up by" language. Five meanings emerged (which we report in Part 2 of the Results section), including each of the meanings for *m* identified in the interviews (which we report in Part 1 of the Results section). We compiled all of the episodes in which slope was formally or informally explored in class. For each category of meaning for *m,* we assembled the relevant classroom episodes and then created and coded categories for the features that appeared to focus students' attention on that particular meaning for *m.* We refer to these as instructional categories to differentiate them from the categories of meaning for *m.* We then made a final pass through the data, making connections across the instructional categories. As a result, the central theoretical construct of focusing phenomena emerged. This article presents a representative example of each of the four focusing phenomena that emerged during our analysis.

RESULTS

Part 1: Students' Generalizations About Linear Equations and *m*

Overview

All of the students who were able to write an equation for a given line (Figure 1) or table (Figure 2) wrote an equation of the form $y = \square \pm \square\, x$. They referred to the value in the first box as the "starting point" and the value in the second box as "what it goes up by." Therefore, they appeared to have generalized a linear equation as "y equals the 'starting point' plus 'what it goes up by' x." One participant was unable to write an equation for the line, and two participants were unable to write an equation for the table. We present the four interpretations that students held for the value of the second box in $y = \square \pm \square\, x$, which we refer to as m for the sake of brevity, although the teacher used the box symbol rather than m in the classroom. We demonstrate that in each case the students appeared to have conceived of m as a difference rather than a ratio and generalized this understanding to the novel interview tasks.

Interpretations of m

Scale of the x axis. When responding to the graphical situation in Interview 1 (see Figure 1), roughly one half of the students identified the value for m as the scale of the x axis. They referred to this value as "what it goes up by," which appeared to mean the interval between successive tick marks along the x axis or "what the x's go up by." For example, Enrique wrote the equation $y = 76 + 2x$ and explained his choice of 2 in terms of the scale of the x axis, "This is the temperature you start at [pointing to 76], then every time one x goes up, every time a different number comes up, it goes up by 2, like, 78, 80, 82 [sweeping his right hand along the x axis from left to right], so it goes up every time by 2." Linnea and Ruben also used 2 as the value for m in their respective equations, $y = 5 + 2x$ and $y = 10 + 2x$, and they explicitly referred to the constant difference of 2 in the numbers along the x axis. A fourth student, Adolfo, also concentrated on the scale of the x axis, although he was unable to produce an equation. He marked the axis with additional tick marks so that it read 76°, 77°, and 78°. When asked to write an equation, he stated, "This one is going by 1s, like this [pointing to the x axis], and I think it has to do something about temperature, like, plus 1." However, he then identified the scale of the y axis as 5 and was not sure which scale to use.

Change in y values. There were four instances in which students interpreted m as the change in y values. In the tabular setting used in Interview 2 (see Figure 2), the dominant strategy for determining the value of m was to identify the difference in y values without regard for the change in corresponding x values. All of the participants expressed difficulty with the table because the intervals were not uniform. For example, Adolfo stated, "I can't figure out a way to tell you what it's going by because the number keeps on getting bigger and bigger; from 2 to 12 it's 10, then it doubles because 12 plus 12 is 24." Although the faucet dripped 10 oz between 7:00 to 8:15, he resolved his dilemma by deciding, "I'll just pretend it was from 7:00 to 8:00." Subsequently, he used 10 oz per hour as the value for m. Linnea and Carissa also determined m by choosing an arbitrary successive pair of y values in the table, calculating the difference, and then using that difference to write an equation. None of these students attended to the corresponding change in x values.

Change in x values. One student, Ruben, originally wrote the equation $y = 10 + 2x$ for his line of best fit for the task shown in Figure 1, choosing the scale of the x axis as the value of m. However, he then indicated that he was unsure about his choice and switched to a different method for determining m. He identified two nonsuccessive points on the graph with x values of 78 and 83, found the difference to be 5, and changed his equation to $y = 10 + 5x$. Therefore, he appeared to view m as "what the xs go by," first as the scale of the x axis and later as the change in x values for two points.

Change in y values with disconnected ratio reasoning. Two students demonstrated some degree of reasoning with the ratio of the change in y values to the change in x values. However, both students ultimately treated m as the change in y values. Enrique represented the other part of his ratio (the change in x values) as the b value in his equation, whereas Carissa appeared to reason only implicitly with the change in x values and did not represent this value explicitly in her equation. We describe each student's work to capture the subtle but important differences in their reasoning.

When asked how quickly the water was leaking given the table of data shown in Figure 2, Enrique correctly determined that 2 oz of water leaked every 15 min. He provided some evidence of ratio reasoning by iterating the 2:15 unit to solve prediction questions. However, when asked to write an equation, he wrote, "T = 15 + 2 oz." Enrique explained his thinking, "So time is for T and 15, that would be like for 15 minutes, and then you add 2 because it's going 2 ounces every 15 minutes." The main idea in his explanation is that 2 oz drip every 15 min. He captured this ratio in his equation by placing 15 in the first box (which he also referred to as the starting point) and 2 in the second box. Although Enrique appeared able to construct a ratio as he wrote the equation, he treated m as the change in y values and b as the change in x values.

Carissa also appeared to demonstrate some limited ratio reasoning, but the evidence is ambiguous. Carissa successfully used the slope formula for the task shown in Figure 1. She located x values of 78 and 80, found their associated y values of 15 and 10, calculated the change in y values as 5, and explained that "it's [meaning the line] going up by 5s." When asked what she would do with the 5, Carissa obtained 2.5 by dividing 5 by 2. Although Carissa could carry out the correct calculations, there is mixed evidence that she reasoned with ratios when she divided 5 by 2. In particular, we would expect a student who had formed a ratio to be able to reason that if 5 more boogie boards are rented when the temperature increases by 2°F, then 2.5 more boogie boards will be rented for every 1°F increase in temperature. In contrast, when asked about the meaning of 2.5 in her equation $y = 8 + 2.5x$, Carissa explained that 2.5 meant "how much it's changing by every time." When questioned about the meaning of "every time," she elaborated that the sales increased by 2.5 boogie boards for every 2°F increase in temperature. It is possible that Carissa misspoke and that every time actually meant 2.5 boogie boards for every 1°F increase. The interviewer returned to this question later in the interview to find out.

When the interviewer asked Carissa to interpret 2.5 again later in the interview, Carissa again maintained that 2.5 more boogie boards were sold for every 2°F increase. Then the interviewer drew Carissa's attention to the apparent contradiction between this interpretation and Carissa's earlier statement that 5 more boogie boards would be rented for every 2°F increase. Carissa responded by changing her equation from $y = 8 + 2.5x$ to $y = 8 + 5x$. Instead of correcting her interpretation of 2.5 to mean 2.5 boogie boards per 1°F increase in temperature, Carissa clearly chose a difference of 5 boogie boards as the value for m. She seemed to know that temperature was important because she stated that 5 more boogie boards were rented for a 2°F increase, but she did not capture the associated temperature value explicitly in her value of m in the equation. Because 2°F corresponds to the scale of the x axis, Carissa could have conceived of m as the amount the line goes up by (i.e., the change in y values) over an implicitly understood interval marked by the tick marks on the x axis.

There is additional evidence to support the conjecture that Carissa viewed m as a difference rather than as a ratio. The interviewer asked if Carissa could substitute

100 for x in the equation $y = 8 + 2.5x$ to determine the number of boogie boards rented at 100°F. Carissa explained that this would be impossible because "that's like the temperature [the 100°F], and this is how many boogie boards they are selling (pointing to the 2.5 in her equation)." Therefore, she identified 2.5 as the number of boogie boards, not as a ratio of boogie boards to temperature, and maintained that she could not multiply boogie boards by temperature.

Enrique and Carissa both appeared to treat m as a change in y values. However, they also seemed to understand that two quantities, the change in y values and the change in x values, were important. For Carissa, the change in temperature was implicit and not represented in her equation. For Enrique, the change in time was explicitly represented in his equation but as the y intercept value. This necessitated the creation of a separate category for their reasoning rather than collapsing it with the change in y values category.

No m value produced despite use of the slope formula. Two students recalled the slope formula but ran into difficulties when using it and were unable to produce a value for m. In Priyani's case, the difficulty seemed to be conceptual in nature. When trying to calculate m for the leaky faucet data (Figure 2), Priyani recited the slope formula as she understood it:

> Ms. R told us that what is changing here [points to the water column] and what is changing here [points to the time column], you divide both of them and then you get a rate of change. And whatever number you get it, you're supposed to put here [points to the second box in $y = \square + \square x$].

However, she could not proceed because the table did not have uniform intervals. Specifically, Priyani stated, "I'm not sure … you see here it's going by 15, and then it's going by 30, and then it's going by 1 hour and 15 minutes, so I don't know how, how to find out what it's going by." For someone who is mathematically sophisticated, $\Delta y \div \Delta x$ represents a corresponding set of y and x intervals that can be normalized to a particular amount of the y quantity per one unit of the x quantity through the operation of division. In contrast, Priyani appeared to conceive of Δy (or Δx) as a set of constant differences of the y (or x) quantity. Consequently, when the intervals were not constant, she could not proceed.

Summary of Part 1

All of the students who were able to write an equation for a given line or table appeared to have generalized a linear equation as "y equals the 'starting point' plus 'what it goes up by' x." Students interpreted the m value in their equations as the scale of the x axis, the change in y values, or the change in x values. Two students demonstrated some ratio reasoning, but neither student explicitly accounted for the change in x values in the value of m. Therefore, all of the interview participants appeared to have formed an understanding of *m* as a difference rather than a ratio and generalized this understanding to the novel interview tasks.

Part 2: How Four Focusing Phenomena Supported the Generalization of *m* as a Difference Rather Than a Ratio

Given that the interview sample was comprised of the higher performing and more diligent students in the class, one might expect that the interview participants would have encountered little difficulty interpreting *m* as a ratio. We were particularly surprised when about one half of the participants treated *m* as the scale of the *x* axis, because the nature of the function is not dependent on the value chosen for the scale of either axis. We therefore sought to explain how the instructional environment afforded each of the categories of meaning for *m* by looking for regularities in the ways that the environment focused students' attention on certain mathematical properties. The central theoretical construct of focusing phenomena emerged from our grounded analysis. Specifically, we identified four major focusing phenomena: (a) the "goes up by" language, (b) the use of well-ordered tables, (c) the ways in which graphing calculators were used, and (d) the emphasis on uncoordinated sequences and differences. This article presents a representative example of each of the four focusing phenomena.

We show how the focusing phenomena regularly directed students' attention to the rate of change as various differences—Δy, Δx, and the scale of the *x* axis—rather than to the coordination of two quantities (see Figure 4). As seen in Figure 4, these four categories interact and affect one another. In particular, the "goes up by" language occurred throughout the entire instructional unit and served to facilitate the attention on differences within each of the other three focusing phenomena.

Focusing Phenomenon #1: "Goes Up By" Language

Ambiguity of the language supports multiple interpretations of m. We coded the classroom transcripts for each occurrence of the "goes up by" language. Five meanings emerged for "goes up by:" (a) the change in *y* values of the function per one unit change in *x* values (what the line goes up by), (b) the change in *y* values of the function (what the *y*s go up by), (c) the change in *x* values of the function (what the *x*s go up by), (d) the scale of the *x* axis (what the *x*s go up by), and (e) the scale of the *y* axis (what the *y*s go up by). Only the first is a mathematically correct meaning for slope; the other interpretations represent differences. We provide evidence that because the teacher and the students used the phrase "goes up by" in five different ways, some students conflated the various meanings. Furthermore, because the teacher associated slope with the first interpretation of "goes up by," some students apparently linked the other interpretations with slope.

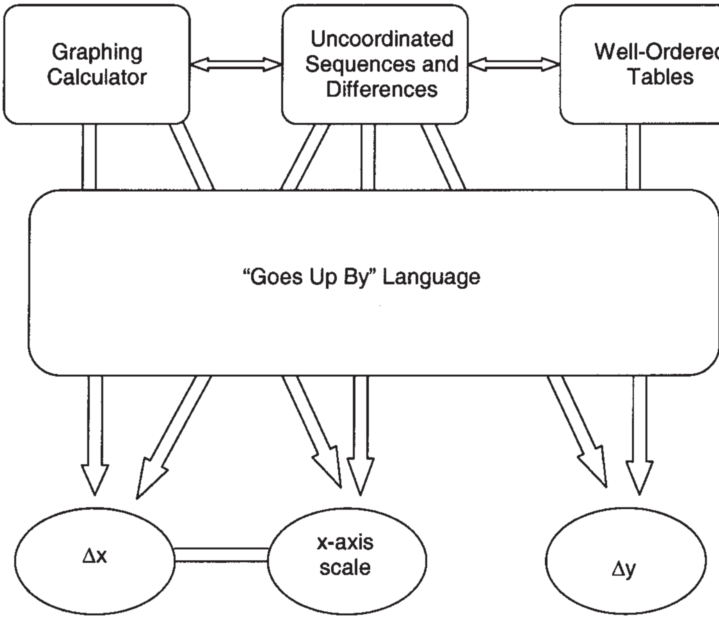

FIGURE 4 Focusing phenomena and their apparent relations to one another and to students' generalizations.

The ambiguity of the "goes up by" language allowed the students and the teacher to talk past one another. For example, the following excerpt from Day 7 demonstrates how the students and the teacher could have used the same phrase to refer to different quantities. The discussion was about a linear graph showing the relation between a weight on a rubber band and its corresponding length:

Ms. R: … and so if you, if I ask you what's the rate of change of this graph, what do they really want to know, do you think?
Salvador: A rate.
Ruben: How much it's going up by.
Ms. R: How much it's going up by. That's really good, Ruben. The rate of change is basically how much is this line going up by, how much it's changing … the rate of change is the change in the length divided by the change in the weight. Because the rate of change is specific to one thing; they want to know how much is it changing or how much is it going up by for every one change in the weight.

The teacher used "goes up by" to refer to a ratio, as indicated by the last sentence of the excerpt. In contrast, Ruben could have interpreted the phrase as the scale of the x axis or the change in x values of the function, as he did during the interviews. Because the students and the teacher could simultaneously hold different meanings for the same phrase, they engaged in discussions in which they never had to clarify specific meanings for rate of change beyond "goes up by."

Focus on the change in y values. We illustrate how the "goes up by" language focused attention on slope as a difference rather than a ratio by examining the focus on the change in y values during the first activity of the unit. Students collected data and developed a table (see Figure 5) depicting an approximately linear relation between the distance of an overhead projector from a wall and the enlargement factor between a real and projected image. Once the second through fourth rows of the table shown in Figure 5 had been completed, the teacher invited the students to look for patterns.

Ms. R: What does that 10.3 tell us then?
Student: It goes up by 3.
Ms. R: It goes up by 3? So you mean this way it goes up by 3 [sweeps hand vertically down the column of y values in the table]?
Student: Yeah.
Ms. R: That looks like a good pattern. Approximately 3? Very good.

This excerpt includes the first instance of the "goes up by" language in the instructional unit. The students first developed this language, and it was later appropriated by the teacher. The teacher's gesture sweeping vertically down the y col-

Distance from Board	Enlargement Factor
1	2.2
2	4.8
3	7.8
4	10.3
5	13
6	15.8

FIGURE 5 Data collected by students showing the enlargement factor of an overhead projector placed varying distances from the board.

umn likely focused additional attention on the y values. When asked to predict the enlargement factor when the projector was 5 m and 6 m from the wall, several students responded that the enlargement factors would be 13 and 16, respectively, suggesting that they attended to the difference between successive y values. Once all the data were collected, Ms. R again invited the students to look for patterns in the table:

Ms. R: Let's see. Sandra, can you tell us what pattern you see in the table?
Sandra: It goes up by like 2 or 3.
Ms. R: OK. So she says that it increases by about 2 or 3. Every how often does it increase by 2 or 3?
Sandra: Every meter.
Ms. R: Every meter. OK. That's good.

Clearly the teacher understood "about 2 or 3" as a ratio because she prompted Sandra to notice how much the enlargement factor increased for each meter. However, that value could have simply represented a difference between y values for Sandra and others. The change in y values and the ratio of Δy to Δx are calculationally the same when $\Delta x = 1$ but not conceptually the same. We elaborate on these conceptual differences, as well as on the significance of this initial focus on the change in y values, when we present an example near the end of the next section in which $\Delta x \neq 1$.

Focusing Phenomenon #2: Well-Ordered Tables

Focus on the change in y values when $\Delta x = 1$. In addition to the "goes up by" language, the use of well-ordered tables encouraged a focus on the change in y values. As described previously, the unit began with the creation of a well-ordered table in which the differences between x values was 1 (see Figure 5). The teacher explicitly directed the students to focus on the change in y values by asking, "So if we took one of the measurements that we have right now, how would we get to the next number, would you add exactly 2 or exactly 3?" The students suggested finding the average of the change in y values, which was 2.7. The teacher summarized the meaning of 2.7 by stating, "To get to the next number, you add approximately 2.7, so 2.7 is how much you do to one number to get to the next number." Therefore, she placed an emphasis on the iterative relation between successive y values in the table.

In addition, the instructional environment supported a connection between the change in y values and the value of the second box in $y = \square \pm \square x$. The teacher explained that students should first write the "starting number," which is the "starting number at the very beginning of the line." She then directed students to focus on the change in y values to complete the equation, as indicated in the following excerpt:

Ms. R: So you're starting at 0. And to get to the next one, what are we doing?
Students: Adding.
Ms. R: Adding. So, I write plus [writes + to show, E = 0 +]. Then how much typically are we adding to get to the next one?
Students: 2.7.
Ms. R: 2.7 [writes 2.7 to show, E = 0 + 2.7]. And how often are we adding 2.7?
Students: Every 1 meter.
Ms. R: Every meter. So what do you want to put after the 2.7?
Students: x.
Ms. R: x [writes x to show, $E = 0 + 2.7x$].

The teacher ended by cautioning the students to "remember what is the starting value and what you are doing to get to the next value." Therefore, she directed the students' attention to the change in y values rather than to the dependency relation between the corresponding x and y values. This was possible because the data were well ordered, with the change in x values being 1. In these cases, the change in y values resulted in the correct slope value.

The teacher saw the significance of the change in x values being 1 because she repeatedly emphasized that 2.7 meant 2.7 for every 1 m. However, the students may not have understood that the value of 1 was significant, because the teacher directed their attention to the 1 m after the 2.7 had already been produced. Some students could have concluded that they should place the change in y values in the second box in a linear equation but interpret that value in terms of the change in x values. For example, if the change in x values had been 3, the students might write the equation with m as 2.7 but then interpret m as 2.7 for every 3 m. This is consistent with Carissa's work in Interview 1, in which she calculated m as the change in y values but interpreted m in terms of some implicitly understood amount of the other variable (presumably the interval between successive tick marks on the x axis).

Focus on the change in y values when $\Delta x \neq 1$. During every class activity involving a well-ordered table in which the change in x values was not 1, it became clear that many students associated m with the change in y values. A typical example occurred during the warm-up for Day 10. Students worked individually to find the rate of change of a function depicted in tabular form. The values in the x column were 0, 3, 6, and 9, and the corresponding y values were 20, 32, 44, and 56. During the class discussion of the problem, the teacher guided a student through the process of calculating the rate of change, reminding him to divide his response of 12 by the change in x values. She noted the widespread response from students that the slope of the function was the change in y values: “A lot of people put 12 as their slope. You’re right that it is going up by 12 when x goes up by 3, but we want to know what it goes up by if x is only going up by 1.”

The fact that the change in y values produces a correct slope when the change in x values is 1, the teacher’s emphasis on this sequence of differences, and the fact that the unit began with a focus on the change in y values, might have set the tone for the entire unit. The focusing phenomena of well-ordered tables could have contributed to the persistence of the conception that the slope was the change in y values. However, as seen in Part 1 of the Results section, many students also interpreted m as either the change in x values or the scale of the x axis. The next two sections detail the ways in which the instructional environment supported a focus on these differences.

Focusing Phenomenon #3: Graphing Calculator

In 4 of the first 6 days of the instructional unit, the use of the graphing calculator and the “goes up by” language supported a focus on differences, specifically the change in x values of a function and the scale of the x axis. The teacher referred to both of these values as what the xs “go up by” in the context of entering values into the calculator. We show that the instructional environment directed students’ attention to the change in x values and to the scale of the x axis. We then demonstrate how some students failed to distinguish between the change in x values and the scale of the x axis, and how these quantities were ultimately conflated with the meaning for slope.

Focus on the change in x values with table setup. The following example illustrates the manner in which the students and the teacher created graphs and tables of linear data with the graphing calculator. To create a table of values for a given function, students first enter an equation and then use the table setup feature shown in Figure 6. To use this feature, students must determine the table’s first x value, called *TblStart,* and the change in x values, called ΔTbl. The students then made predictions with the table feature. For example, in the following excerpt from Day 2, the students typed in the equation $y = 2.7x$. The teacher showed the class how to use the table setup feature to find the corresponding y value when x was 2.5 by creating a table that included 2.5. Her use of the phrase “going by” to re-

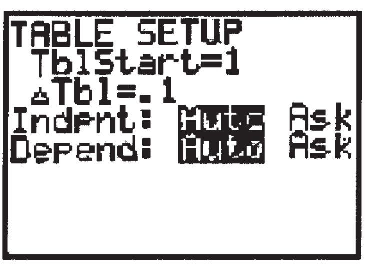

FIGURE 6 TI-83 commands for creating a table.

fer to ΔTbl, and her emphasis on using the table setup feature encouraged the students to attend to the change in *x* values:

> What could we put on table setup to get that to come out? That was the question. Going by .5s? And you could start it where? You could start at 0 [referring to the TblStart input] going by .5s [referring to the ΔTbl input] if you guys want to.

Focus on the x axis scale with the window function. The only other calculator function the students and teachers used in this instructional unit was the window function, which allows users to specify the domain and range of graphs. After the students entered data or an equation into their calculators, they used the window function to input the minimum and maximum values for *x, y,* and the scale of each axis (see Figure 7). In the following excerpt, the teacher demonstrated how to recreate the graph shown in Figure 8. She entered 0 for Xmin and 80 for Xmax. The next prompt was Xscl, which refers to the scale of the *x* axis:

Ms. R: And *x* scale [points to Xscl function on the overhead calculator display] means, what are they counting by. Look at the graph on your paper—What are they counting by on the bottom?

Student: 10.

The teacher used the phrase "counting by" to refer to the scale of the *x* axis. It is possible that the use of "counting by" rather than "going by" might have helped the students distinguish the scale of the *x* axis from other constructs "going by" referred to. However, the students used "going by" to refer to scale of the *x* axis, which suggests that they did not distinguish between the two phrases.

The graphing calculator requires the user to input a value for the scale of the *x* axis before viewing a graph or using a table. The teacher's attention to this requirement further highlighted the importance of considering the scales on the *x* and *y* axes. She encouraged the students to articulate a value for the scale of the *x* axis for every calculator-related problem. Therefore, the instructional environment emphasized a focus on the scale of the *x* axis to a degree that might not have occurred without the use of the calculator.

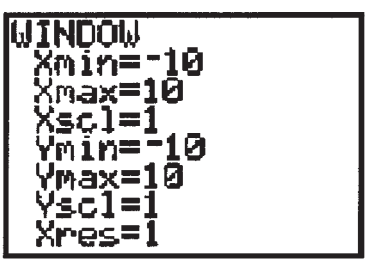

FIGURE 7 TI-83 commands for creating a graph.

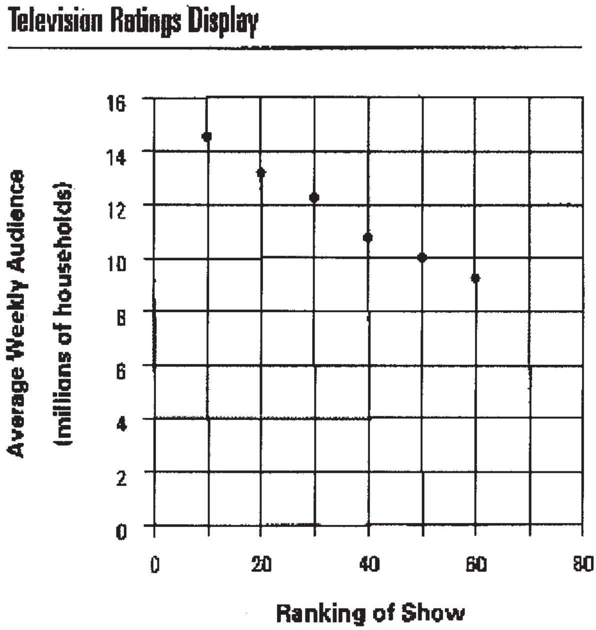

FIGURE 8 Graph showing the relation between the ranking of a television show and the audience size for the show. From *Contemporary Mathematics in Context, Course 1 Part A* (p. 162), by A. F. Coxford, J. T. Fey, C. R. Hirsch, H. L. Schoen, G. Burrill, E. W. Hart, et al., 1998, Chicago, IL: Everyday Learning Corporation. Copyright 1998 by The McGraw-Hill Companies. Reprinted with permission.

Conflation of Δx and the x-axis scale. The following excerpt demonstrates that the students in the target group appeared to conflate the change in x values with the scale of the x axis. Students used the calculators to create a scatterplot of the data in Figure 9, showing the relation between predicted concert audience size and ticket price. Enrique helped Marina determine what to enter for each calculator prompt in the window function (Figure 7). They successfully entered minimum

Price	5	7.50	10	12.50	15	17.50	20
Attendance	210	175	155	135	113	70	50

FIGURE 9 Data showing predicted concert audience sizes for various ticket prices. Adapted from *Contemporary Mathematics in Context, Course 1 Part A* (p. 176), by A. F. Coxford, J. T. Fey, C. R. Hirsch, H. L. Schoen, G. Burrill, E. W. Hart, et al., 1998, Chicago, IL: Everyday Learning Corporation. Copyright 1998 by The McGraw-Hill Companies. Adapted with permission.

and maximum x values for the x axis of their graphs. Then Enrique, looking at the Xscl prompt on the calculator, asked Marina, "What are we going up by?"

Enrique: What are we going up by?
Marina: It's going by 5, 6, 7, 7.5, oh, it's going up 2.5.
Enrique: No.
Marina: Yes it is.
Enrique: What, what, what, what?

Enrique used "going up by" to refer to the scale of the x axis. However, Marina responded by referring to the change in x values in the table. The students' use of "goes up by" in combination with their calculator usage helped support an association between the scale of the x axis and the change in x values of the function. Furthermore, because both constructs involve x values, it is reasonable that some students viewed each as "what the xs go by." The commands ΔTbl and Xscl were mediated by the same physical device. This, along with the "goes up by" language, may have contributed to an inappropriate conjoining of the two constructs, particularly because the table set and window functions were the only nonarithmetic calculator functions used in the instructional unit. In addition, when creating a graph from a well-ordered table, it is natural to use the Δx value as the parameter for Xscl, further linking the two. However, students may not have realized that the x values of a function are independent of the scale of the x axis, and that ΔTbl can always be chosen so that it differs from Xscl. Finally, the same language referred not only to the x-axis scale and the change in x values, but also to the value of m. In the next section, we show that the students ultimately conflated these differences with the meaning for the slope of a linear function. This conflation could have supported students' interpretations of m as the scale of the x axis, as reported in Part 1 of the Results section.

Focusing Phenomenon #4: Uncoordinated Sequences and Differences

The teacher introduced uncoordinated sequences as a way of helping students manage the difficulties of calculating m in well-ordered tables with $\Delta x \neq 1$ and in graphs with the x axis scale not equal to 1. The teacher and the students engaged in three types of activities that employed uncoordinated sequences of numbers: (a) determining the slope by creating uncoordinated sequences in a table, (b) linking the slope formula to two uncoordinated sequences and differences in a table, and (c) linking the slope formula to two uncoordinated differences in a graph. In each case, students' attention became focused on differences instead of on the ratio between the changes in corresponding y and x values. In the first case, the students and teacher focused on the change in y values. In the second and third cases, they focused on Δy and

Δx in the slope formula as uncoordinated differences. In the third case, they also focused on the scale of the x axis. We describe each case in turn.

Focus on the change in y values through the creation of uncoordinated sequences in a table. On Day 3, the students first encountered a well-ordered table in which the change in x values was not 1 (see Figure 10). The table represented another linear relation in the overhead projector situation, namely the relation between the length of any line placed on the projector (sketch length) and the length of the projected image (screen image length).

As the students tried to complete the table, they initially focused on the change in y values. Specifically, Jerry remarked, "So what's it going by? 0, 6, 12, 18, … ." Enrique noted, "It looks like it's going by 6." One student, Soledad, noticed that she could also multiply x by 3 to obtain a corresponding y value, but the other group members ignored her and continued to find a way to use 6 in their equations. The following excerpt shows that the teacher responded by encouraging the students to write the "in between" values in both rows so that the new table would increase in increments of 1 for x:

Ms. R: Remember when we're describing a pattern it's best if we describe the pattern for what happens when you add just 1 unit extra to it. So instead of adding 2 centimeters, what do you think the new pattern would be if we put in the missing numbers in between [writes 1 between 0 and 2 and writes 3 between 2 and 4 in the top row of the table]? How much would it be going up by here [points between the 0 and 6 in the bottom row]?

Salvador: 3.

Ms. R: By 3. Salvador says 3. Does that make sense?

Jerry: Yes.

Ms. R: What would this one be right here [points between 6 and 12]?

Complete this table in a pattern that you would expect for other data pairs.

Sketch Length (cm)	0	2	4	6	8	10	12	14	16	18
Screen Image Length (cm)			12			30				

FIGURE 10 Data indicating a relation between the length of line segments drawn on an overhead transparency and their projected images. Adapted from *Contemporary Mathematics in Context, Course 1 Part A* (p. 161), by A. F. Coxford, J. T. Fey, C. R. Hirsch, H. L. Schoen, G. Burrill, E. W. Hart, et al., 1998, Chicago, IL: Everyday Learning Corporation. Copyright 1998 by The McGraw-Hill Companies. Adapted with permission.

Andy: 9.

Ms. R: 9. What would this one be in between here [points between 12 and 18]?

Enrique: 15.

Ms. R: So if we were going up every 1 centimeter, and making each picture 1 centimeter longer, how much would the bottom number go up by?

Students: [No response.]

Ms. R: 3, right? Okay, so in this case for 2 centimeters we saw it's going up 6, but if we were just looking at 1 instead of 2, it looks like it's going up by … How many?

Students: [No response.]

Ms. R: For 2 it was 6, for 1 it's … ?

Students: 3.

Ms. R: So the equation will be I equals 3 times S [writes I = 3S].

The teacher's direction to fill in the "in between" numbers for both rows in the table meant that the students could then use the change in y values as the m value of an equation. Ms. R's language "for 2 it was 6 and for 1 it's 3" indicates that she conceived of ratios in the table. However, the students could have interpreted the new table as representing two uncoordinated sequences of numbers. The instruction and the co-constructed use of "goes up by" could have contributed to the focus on two uncoordinated sequences of numbers. In particular, the teacher used the "goes up by" language to note that the numbers increased by 1 in the top row and by 3 in the bottom row. Addressing each sequence separately may have contributed to a lack of coordination of the two sequences. Finally, once the students had created a new table, they could write the equation I = 3S without forming a ratio for the slope value. The aforementioned method grew into a standard mathematical practice for determining slope values from tables of data, and thus continually reinforced a focus on differences rather than on ratios.

Focus on Δy and Δx as uncoordinated in the slope formula. The teacher introduced the $\frac{\Delta y}{\Delta x}$ formula for determining the rate of change of a linear function on Day 7. She later established the slope formula as $\Delta y \div \Delta x$ when she defined slope as a constant rate of change on Day 9. On Day 8, students practiced calculating the rate of change for a linear relation between the number of ounces hung on a spring and its corresponding length (see Figure 11). To prompt the students to calculate the rate of change, the teacher asked them to determine how much the spring would stretch if 1 oz were added. Several students called out "10," which suggests that they once again attended to the change in y values without considering the change in x values.

Weight (in ounces)	Length (in inches)
0	10
5	20
10	30
15	40
20	50
25	60

FIGURE 11 Table of data created by the teacher involving a relation between the number of ounces of weight hung on a spring and the length of the spring.

The teacher responded by demonstrating two methods to determine the rate of change. First she created two uncoordinated sequences, using the method described in the previous section of this article. She filled in the whole numbers between 0 and 5 in the weight column and asked students to fill in the corresponding spring length values. Second, the teacher reminded the students of the formula for rate of change from the previous lesson:

> Remember the rate of change was the change in length divided by the change in the weight. The length changed from 10 to 20 by how much? By 10. And the weight changed from 0 to 5, which is 5. So 10 divided by 5 is 2. So you have two options. If you guys want to make the table go by 1s instead of 5s, you can do that, and then you can figure out the numbers here. Another way you can do it is just say, OK if it's going up by 10 here [sweeps hand down the column of length values in the table] and it's going up by 5 here [sweeps hand down the column of weight values in the table], then 10 divided by 5 is 2.

By saying "if it's going up by 10 here" and gesturing down the column of length values, the teacher directed the students' attention to a sequence of y values of the function with a constant difference. She similarly directed their attention to the sequence of x values. This may have supported the conception that the change in y values (or the change in x values) represents a constant difference in a sequence of uniformly increasing or decreasing values. In contrast, Δy should represent an interval between any two y values of a linear function and Δx the interval of corresponding x values.

This approach could help explain Priyani's behavior in Interview 2 when she was asked to write an equation for the data in Figure 2. She successfully recited the slope formula, but could not proceed because she did not know "how to find out what it's going by." "What it is going by" appeared to mean the constant difference in a sequence of uniformly increasing or decreasing values. Therefore, Priyani could not determine Δy because the change in y values was not constant. The manner in which the teacher linked Δx and Δy to two sequences of uncoordinated x and y values likely supported this conception.

Furthermore, the instructor stated rather than developed the notion that dividing Δy by Δx will produce a ratio of the difference in y values per 1-unit change in x values. If students do not have sufficient understanding of the meaning of division, then dividing two differences may not necessarily produce a ratio for the students. The instructor directed students' attention to two differences linked by the numerical operation of division, which may have contributed to a conception that the formula simply adjusts the change in y values, leaving slope a difference rather than transforming it into a ratio.

Focus on scale of x axis and change in x values when determining slope from a graph. In the graphing calculator section, we argued that the graphing calculator and the "goes up by" language supported the conflation of the scale of the x axis with the change in x values for some students. In this section, we describe how the instructional environment could have supported the linkage of these two differences with the slope of a line. On Day 8, students worked in small groups to calculate the rate of change of each line shown in Figure 12. The teacher directed students to first create a table of values for each line by using the scale of the x axis as the change in x values. She may have further contributed to a conflation of the change in x values with the scale of the x axis by referring to both with the "goes up by" language:

> Be careful. Are they going by 1s down here on this graph [sweeps hand from left to right along x axis of graph]? What are they counting by on the bottom? By 10s, right? It's 0, 10, 20, 30, so the tables on your paper are gonna go by what? By 10s; so be careful when you're doing your tables that you go by 10s and not by 1s anymore.

Later in the same lesson, the teacher used similar language to refer to the rate of change of the line, "If you wanted the rate of change, you want to see how much the lines are going down by." The ambiguity of this language could have served as an additional support for the link between the scale of the x axis, the change in x values, and the rate of change or slope of a line.

The teacher also regularly drew students' attention to the x axis through the use of a visual representation of the change in x values on a graph. She demonstrated the same method for finding the slope of a line nine times over the course of the unit. The following example from Day 8 is typical. The teacher demonstrated how to calculate the slope of line D in Figure 12. She drew points on the line at (10,8) and (30,4), circled the y value of each point on the y axis, and showed the students

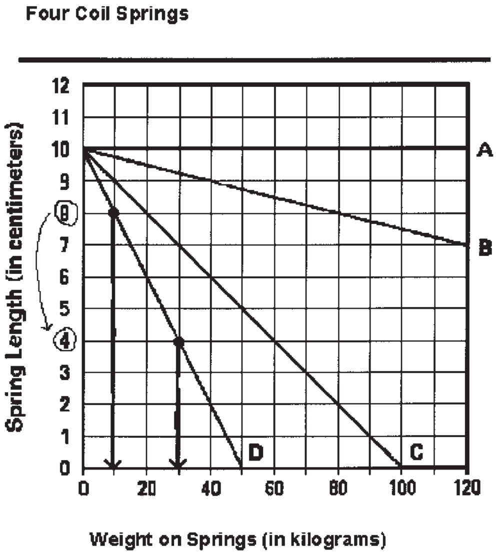

FIGURE 12 Visual representation used by the teacher to calculate the slope of a line. From *Contemporary Mathematics in Context, Course 1 Part A* (p. 185), by A. F. Coxford, J. T. Fey, C. R. Hirsch, H. L. Schoen, G. Burrill, E. W. Hart, et al., 1998, Chicago, IL: Everyday Learning Corporation. Copyright 1998 by The McGraw-Hill Companies. Reprinted with permission.

that the change in y values was 4. She then directed the students' attention to the x values by drawing vertical lines from the points to the x axis. The students calculated the change in the two x values to be 20, and divided 4 by 20 to obtain 0.2 for the slope. The teacher summarized by explaining that they must always "find two points, see how much the length changed, and divide that by how much the weight changed; that's how you use the graph to come up with the rate of change."

The instructional treatment of Δy and Δx as separate differences again supported a focus on two uncoordinated quantities. The teacher stated that the length of the spring "dropped down by" 4 when she calculated Δy, and explained that 20 was the amount of weight added on when she calculated Δx. At no time did she explicitly coordinate these quantities, for example as 4 cm for every 20 kg. Rather than treating the relation between Δy and Δx as a ratio, she treated the two quantities as uncoordinated differences. This supports the possibility that the slope formula represented two differences for students, linked by the numerical operation of division, which may simply have resulted in an adjusted difference rather than a ratio. Without a firm association between slope and, in this case, a ratio as a composite unit of length and weight quantities, students may have treated the quantities as isolated and thus might be likely to remember only one of the quantities when calculating slope.

Furthermore if Δx was conflated with the scale of the x axis, it is plausible that the scale of the x axis became associated with slope for some students. The teacher's method of drawing vertical lines to the x axis may have directed students' attention to the x axis during the calculation of the change in x values, further linking the x axis scale with the change in x values. In fact, in the earlier example, the change in x values was the same as the numerical scale along the x axis. Finally, on Day 9, the teacher summarized the method for calculating the slope of a line by placing the symbol Δx along the x axis, as shown in Figure 13. Therefore, the slope formula $\frac{\Delta y}{\Delta x}$ could have become linked to the scale of the x axis for those students who thought that Δx was the scale of the x axis.

DISCUSSION

The results of this study demonstrate that the students formed an understanding of slope on the basis of single quantities changing and generalized that understanding to novel situations. More important, the students' generalizations appeared to be linked to four focusing phenomena, which regularly directed their attention to various sets of differences. We conjecture that by altering the nature of the focusing phenomena in a classroom, we can significantly affect the nature of students' generalizations. Specifically, if one regularly directs students' attention to the coordination of covarying quantities, then students are more likely to generalize slope as a ratio. This curricular recommendation emerges from the theoretical connection between the processes of abstracting and generalizing and from the empirical find-

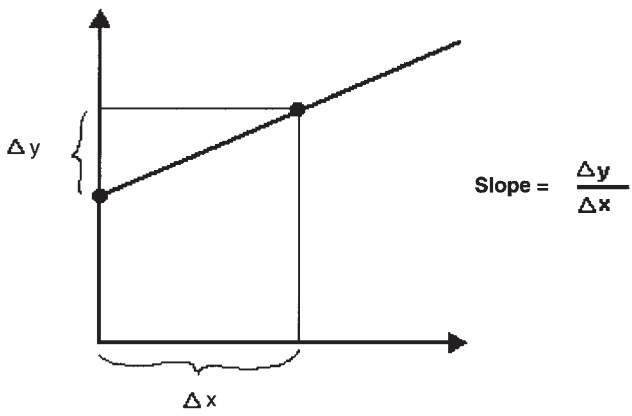

FIGURE 13 Visual display used by the teacher to show a general method for finding the slope of a line.

ings of this study. Both the theory and the data suggest that by changing the nature of the focusing phenomena, we can change the nature of students' generalizations. We do not mean to suggest that redirecting students' attention will result in the automatic or straightforward formation of correct conceptions. Furthermore, the construction of focusing phenomena is not completely under the teacher's control but involves listening to and guiding students while they develop, test, and revise their own ways of thinking. The process of altering focusing phenomena is complex and involves at least four interrelated issues, which we discuss next.

Changing the Object of Focus

We need to do more than simply replace the focusing phenomena documented in this article with new ones; we need to first change the object of focus. Specifically, the object of focus must shift from the production of a slope value to the development of slope as a conceptual entity. Two ways to mentally form a ratio—as a multiplicative comparison or as a composite unit—were described previously in the Conceptual Analysis of Slope section of this article. Consequently, we can evaluate potential instructional alternatives by examining whether the mental formation of ratio is likely to be supported. We consider two examples.

First, the use of the "goes up by" language is arguably the most prominent feature of the instructional environment reported here. Therefore, an attractive way to address the problem that occurred in this classroom is to replace the ambiguous language with more formal mathematical language. However, if the language of m as $(y_2 - y_1) \div (x_2 - x_1)$ or even m as "rise divided by run" is linked with uncoordinated sequences or uncoordinated differences, then similar problems are likely to arise as those reported in this article. Therefore, language use alone is unlikely to fully address these concerns. One difficulty with the "goes up by" language was that it allowed speakers to have different referents and thus to talk past each other. Whereas the language of coordinate pairs in $(y_2 - y_1) \div (x_2 - x_1)$ may be less ambiguous, it may also lack a referent for students and represent nothing more than a computational recipe. In fact, research has documented that students in classrooms that use more formal language still have difficulty thinking of slope as a ratio (Barr, 1981; Leinhardt, Zaslavsky, & Stein, 1990; Lobato, 1996, 2002).

Second, one might advocate replacing the focusing phenomenon of uncoordinated sequences by drawing students' attention across rows in a table with the question, "What do you need to multiply x by to get y?"; or when the y intercept is not zero, "What is the rule that gets you from the x value to the y value?" Although this may be more effective in drawing students' attention to the relation between corresponding x and y values, we believe this is insufficient to support the development of slope as ratio for many students due to the calculational nature of the probe. In particular, the instructional question does not necessarily draw attention to the quantities in the situation. For example, developing the enlargement factor as a conceptual entity in the overhead projector activity (see Figure 10) involves mentally asking how much larger is the length of each projected image than the length of the sketch. Because it is reasonable to compare a sketch length with its projected image length either additively or multiplicatively, it is nontrivial for students to realize that only the multiplicative comparison will hold for all sketch lengths. Simply asking students what number can be multiplied by x values to obtain y values fails to adequately connect the numerical values to the quantities they represent and does not allow students to consider whether an additive comparison between quantities may also hold.

Problematizing the Formation of Ratios

Students need to encounter problem situations that necessitate or problematize the mental formation of a ratio. This did not occur in Ms. R's classroom, because it was possible to negotiate the tasks by operating with differences only. Brousseau (1997) described the Topaze Effect in which teachers attempt to achieve the optimum meaning for the greatest number of students by choosing easier and easier questions. If the target knowledge disappears completely, the Topaze Effect occurs. This also occurs when problem tasks are chosen such that students are deprived of the opportunity to develop, test, and revise the target knowledge. Because the text did not present data in tables in which the independent variable occurs in nonuniform intervals, the students never needed to build an understanding of slope as a ratio between the change in y values and the corresponding change in x values. Students lacked the opportunity to examine whether situations are linear even if the data occur in nonuniform intervals. Problematizing requires the creation of tasks that present a need for students to develop the target knowledge. It is compatible with Harel's (2001) Necessity Principle, which asserts that students are most likely to learn if they see an intellectual need for what we intend to teach them.

A task that might problematize the formation of ratio is the leaky faucet situation (shown in Figure 2) along with the question, "Is the faucet leaking faster at times, or is it leaking steadily the entire time?"[2] The solution to this problem necessitates the coordination of two covarying quantities. Because the data are presented in nonuniform time intervals, the students can no longer attend solely to differences. To answer the question, students need to explore the dependency relation between y and x, and to determine linearity based on a ratio remaining constant throughout the data.

Alternative Focusing Phenomena

As shown in the Results section, focusing phenomena emerge not only through the instructor's behavior but also through students' behavior, curricular materials, arti-

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[2]We are not claiming that this task is a real-world problem because the data are unrealistically neat and do not reflect any measurement error. Furthermore, we are aware there is an implicit assumption that the faucet leaked at a constant rate throughout any given interval.

facts such as graphing calculators, and shared mathematical language. Focusing phenomena are dynamic co-constructions and as such can not be scripted in advance. Therefore, the following five potential focusing phenomena are not fully specified but rather are presented to convey the spirit of how students' attention could be directed toward the coordination of covarying quantities:

1. Collect and display data to support exploration of relations between quantities (not just number patterns). The instructional environment documented in this study directed students' attention away from cause-and-effect relations and toward iterative "goes up by" numerical patterns by compiling data in well-ordered tables, using only y values to make predictions, using the "goes up by" language to ambiguously refer to both a ratio and a difference, and failing to problematize the exploration of cause–effect relations. The cause–effect nature of linear relations is not inherent in real-world situations but requires focus and attention. One way to focus attention on the coordination of quantities is to provide or collect data so that there are not uniform intervals between successive values of the independent variable. For example, using the spring setting from CPMP (see Figure 11), the teacher could present weights such as 5 g, 17 g, 20 g, and 28 g, and challenge students to experiment to determine if there is a relation between the stretch of the spring and the amount of weight hung on the spring. Using the results of experimentation to predict and then test the stretch of the spring for various weights could further focus attention on the cause–effect relation between quantities.
2. Use the language of quantities. The "goes up by" language is a language of recursive number patterns. In contrast, the language of quantities and units can focus attention on ratios. Teachers should insist that any time students speak of a number within a real-world situation, they speak of the quantity for which the number is a value. Furthermore, when students use an arithmetic operation, they should name the quantities on which they are operating, the quantity their calculation evaluates, and what the operation accomplishes in the situation. For example, in the bed springs example (Figure 12), students were asked to write an equation for Line D. Students should be expected to explain that (0,10) indicates that the bed spring is 10 cm long when there is no weight on the bed. They should also understand that the b value in $y = \mathrm{b} + \mathrm{m}x$ represents the value of y (which is the length of the spring) when x (which is the weight placed on the bed) is zero. The point (10,8) indicates that the bed spring compresses until it is only 8 cm high when a 10 kg weight is placed on the bed. These two points could be used to determine that the spring compresses 2 cm for each 10 kg weight that is placed on the bed. By finding $2 \div 10$, one determines the length that the spring is compressed when a 1 kg weight is placed on the bed. Therefore, the slope value of 0.2 cm per kilogram represents a measure of the stiffness of the spring.
3. Use "sameness" activities to form coordinated pairs. The repeated attention on uncoordinated sequences apparently led some students to generate an under-

standing of slope as a difference. The following type of activity can be used to help students form coordinated pairs of x and y values and generate an understanding of slope as a representative member of an equivalence class of ratios. One could ask students to consider that 4 oz of water were collected after 10 min from a leaky faucet. If the faucet continues to drip at the same speed, what other data might be reasonable to collect? The use of pictures and explanations can play an important role in shifting the focus toward reasoning quantitatively with ratios. For example, a student might draw a picture showing 4 oz in a cup with 10 min written next to it and reason that if another 10 min passes, then 4 more ounces should drip because the speed of dripping has not changed. Therefore, the data point of 8 oz in 20 min represents the same dripping speed. An equivalence class of ratios, each representing the same dripping speed, can be developed by using a composite unit ratio of 4 oz:10 min in conjunction with iterating, partitioning, and the idea that a/b of 4 oz:10 min represents the same speed of dripping as a/b of 4 oz in a/b of 10 min. Any member of the equivalence class can represent the entire class and is the slope of the function. These types of explanations preserve reasoning with quantities and form the foundation for conceiving of slope as a measure of the dripping speed.

4. Relate the numerical operation of division to mental operations on ratios. This article reports that division was treated as a numeric operation linking two uncoordinated differences. In contrast, division could be connected with mental operations on ratios. For example, in the leaky faucet situation described in Item 3, the teacher could ask students to draw a picture to show how much water is in the container after 1 min of dripping. Students could partition the ratio 4 oz:10 min into 10 equal parts, each representing 0.4 oz in 1 min. The operation of partitioning could then be meaningfully connected with the numeric operation of division to help students understand that the resulting quotient is a ratio of ounces per minute.

5. Delay the use of graphing calculators. We believe that the students who participated in this study may have benefitted from learning to coordinate two quantities and to create a ratio describing how one quantity changed in relation to the other prior to naming the ratio as slope and prior to representing slope with a graph or equation. We accept the argument that the relation between learning and symbolizing is reflexive and that students may develop an understanding of ratio while they develop an understanding of the Cartesian coordinate system (Sfard, 2000). However, students operating at unsophisticated levels of proportional reasoning might benefit from developing an understanding of the linear functions in real-world settings and from using their own diagrams and symbols to represent the rates of change prior to the introduction of the conventional representations of graphs and equations.

Beyond Slope

Focusing phenomena provide a powerful way to establish connections between students' generalizations about slope and the instructional environment. Although there are additional factors involved in the interplay between students' generalizations and the instructional environment, such as students' prior knowledge, the examination of focusing phenomena in this study offers at least one plausible explanation for the students' generalizations. One could similarly examine the focusing phenomena that afforded students' generalizations regarding the *y* intercept in $y = \square \pm \square x$. Focusing phenomena can also be leveraged to provide alternative instructional ideas to teachers and designers of instructional materials. Finally, focusing phenomena may be widely useful in the establishment of links between specific features of instruction and students' generalizations for many other mathematical topics. Therefore, this article is relevant beyond the teaching and learning of slope. It emphasizes the need to pay greater attention to the specific ways in which teachers, instructional materials, and artifacts direct students' attention when learning any new topic.

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