Patterns, & Linear

Students explore linear functions through patterns and the measurement of real-world quantities.

Amy B. Ellis

Pattern generalization and a focus on quantities are important aspects of algebraic reasoning. This article describes two different approaches to teaching and learning linear functions for middle school students. One group focused on patterns in number tables, and the other group worked primarily with real-world quantities. This article highlights the different learning dynamics, reasoning activities, and student ideas that can emerge in classrooms that take a number-pattern approach or a quantitative reasoning approach to linear functions.

WHAT IS REASONING WITH QUANTITIES?

A classroom teacher asked one student, Sarah, to determine how fast her classmate, Julio, walked. Sarah timed Julio with a stopwatch as he walked at a steady pace. She found that he had walked for 6 seconds and had traveled 15 feet. Sarah knew that one way to measure speed was to measure how far Julio walked in a given amount of time, so she decided that 15 feet in 6 seconds was a good measure of his speed. Another classmate, Ben, said that walking 15 feet in 6 seconds would be the same speed as walking 5 feet in 2 seconds, because you could divide both the feet and the seconds by 3. Julio argued that it would also be the same speed as 2.5 feet in 1 second, because you could just divide both by 6. The students decided that all of these measures described Julio's speed.

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Quantities, Functions

Sarah's measures of distance (15 feet) and time (6 seconds) are examples of quantities. A *quantity* is the measure of some quality, such as distance or time. The quantity consists of the quality (distance), the appropriate unit of measure (feet), and a numerical value (15 feet). When Sarah and her classmates thought about speed as a ratio of feet to seconds, they were reasoning with quantities and their relationships; this type of reasoning is characterized as *quantitative reasoning*. One way to promote algebraic generalizations is to encourage students to engage in quantitative reasoning (Steffe and Izsak 2002; Thompson 1994). The next sections examine different instances of quantitative reasoning in more detail.

Fig. 1 Tasks that involve linear patterns

Task 1
In class, you made the following table of values:

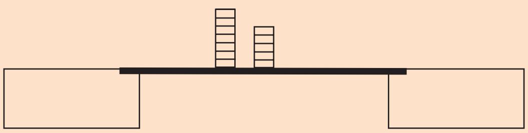
x	*y*
0	0
1	73
2	146
3	219
4	292
5	365

A. What sorts of patterns did you notice in class, and what do you notice now?

B. Do you think this pattern could continue?

C. Can you make additional entries in the table?

Task 2
In class, you made paper bridges to see how many pennies they could hold.



1. What determines how much weight the bridge can hold?
2. Does it matter how long the bridge is? Would a long bridge hold more pennies, or would a short bridge hold more pennies?
3. What is the relationship between the layers of paper and the pennies on the bridge?
4. Why happens if you add an extra layer of pennies to the bridge?
5. Will this relationship always hold, or does it matter what numbers you use?
6. How many pennies would a bridge with 25 layers be able to hold?
7. You have 115 pennies that you want to put on a bridge. How many layers would your bridge need?

UNITS ON LINEAR QUANTITIES

My research involved studying the generalizations of middle school students in different learning and teaching environments. One group consisted of seven eighth-grade algebra students who studied linear functions through a number-pattern approach. The class had a twelve-session unit on linear functions, and each session lasted one hour and fifty minutes. These sessions primarily used number tables to develop generalizations about linear growth. Seven of the thirty-four students in the class agreed to participate in a one-hour individual mathematics interview in which they solved problems and made generalizations. The interview problems were tailored each day to follow up on the tables and the situations that the students had seen in class. An example of two typical interview problems is shown in **figure 1**.

Group 2 consisted of seven prealgebra seventh graders who participated in a fifteen-session teaching unit. Each session lasted one hour.

AN EXTENSION FROM FIGURE 1

For an extension, consider **figure 1**'s question A:

What sorts of patterns did you notice in class, and what do you notice now?

If students notice a recursive pattern, such as *y*s increasing by 73, you can ask the following questions:

- Why does *y* increase by 73 each time?
- Does it matter what the *x*-values are?
- Is there a relationship between the 4 and the 292?
- Does your pattern always hold?

Throughout the unit, the students encountered two real-world situations involving linearity: gear ratios and constant speed. The students first worked with physical gears that they could manipulate to solve problems and later used a computer program called SimCalc MathWorlds® (Roschelle and Kaput 1996), which simulated speed scenarios by showing two characters walking across the screen at constant speeds.

The students were introduced to the idea of gear ratios after working with the Connected Gears problem **(fig. 2a)**. After finding ways to keep track of the gear's revolutions, students were asked to find how many times the small gear would turn if the large gear turned, for example, 120 times. The students then worked with gears of differing sizes and were encouraged to generalize their ideas about the relationship between the rotations of any two gears. When the students moved on to the topic of speed, they worked with tasks such as the Frog Walking problem **(fig. 2b)**.

Table 1 shows how often the students in each group generalized in particular ways. The eighth-grade students focused more on patterns, and the seventh-grade students focused on relationships. The next section provides examples of how the students in each group reasoned and generalized, presenting some examples of the learning dynamics that emerged in the two classes.

Fig. 2 Gears form the basis for the task in (a); a frog's gait is discussed in (b)

The Connected Gears Problem

You have two gears on your table. Gear A has 8 teeth, and gear B has 12 teeth. Answer the following questions.

1. If you turn gear A a certain number of times, does gear B turn more revolutions, fewer revolutions, or the same number? How can you tell?

2. Devise a way to keep track of how many revolutions gear A makes. Devise a way to keep track of gear B's revolutions. How can you keep track of both at the same time?

(a)

The Frog Walking Problem

The table shows some of the distances and times that the frog traveled. Is it going the same speed the whole time, or is it speeding up or slowing down? How can you tell?

Distance	Time
3.75 cm	1.5 sec.
7.50 cm	3.0 sec.
12.00 cm	4.8 sec.
15.00 cm	6.0 sec.
40.00 cm	16.0 sec.

(b)

Table 1 Generalizing in particular ways

Students' Reasoning and Generalizations	Eighth Graders	Seventh Graders
Searching for relationships	7	48
Searching for patterns	21	15
Statements of continuing phenomena	3	50
Statements of global rules	41	26

A CLASSROOM FOCUS ON NUMBER PATTERNS

The eighth-grade students learned about linear functions in a classroom environment that emphasized different number patterns. The students focused on patterns rather than relationships, and their generalizations were typically statements of global rules. Mario was interviewed, and his work with a number table **(fig. 3a)** is an example of searching for patterns. Tracing his finger down the y column, he identified a pattern in the table. He said, "On the x side, it's going up by ones, and on the other side, it's going up by . . . sevens." The students who searched for patterns often found them within a column, rather than focusing on the relationship between corresponding (x, y) pairs in a table.

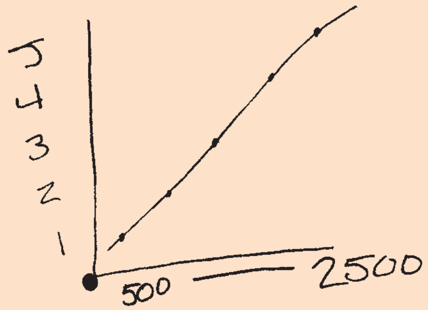
Their focus on recursive patterns in the columns meant that the students developed incorrect global rules about linearity that were dependent on uniform data tables. An example of such a global rule is Juliana's statement about the number table posed during her interview **(fig. 3b)**: "If it's in a continuous pattern that's the same every time, it comes out to be a line." Global rules represent a student's more general meaning for a concept, such as what constitutes linearity or what slope means. The eighth-grade students

Fig. 3 Examples of students' work

x	y
0	5
1	12
2	19
3	26
5	40
6	47

(a)
Mario's number table

X	Y
0	400
1	900
2	1400
3	1900
4	2400
5	2900



(b)
Juliana's number table and graph

developed these types of global rules as they tried to make sense of the number tables and patterns they had found.

When Juliana was asked to justify her global rule, she made a graph to show that it worked **(fig. 3b)**. She struggled, however, to explain why a continuous pattern meant that the data would make a line, saying, "If they're all the same, it's continuously, then it's just gonna be like a little square. You just follow it." Growing frustrated with being unable to clearly explain her thinking, Juliana looked over all the different number tables she had seen in class. She came up with a more strict global rule: "For it to make a straight line, there has to be a pattern this way [gesturing down the *x* column], a pattern this way [gesturing down the *y* column], and a pattern going back and forth right here [gesturing across the columns]."

Although Juliana's new global rule was not correct for all tables, it made sense to her because she had never found any patterns in number tables that did not increase in even increments on the *x* column (i.e., *x* values increasing by 1 each time, or by 5 each time). Because those are the only tables that Juliana had ever identified as being linear, she believed that any set of data must have three patterns to make a line. The only explanation she could provide, however, was to point to several examples that showed the patterns in question. Because she had learned about linear functions in a classroom environment that heavily emphasized one particular type of table or pattern, it made sense that she would use only those tables when trying to justify her rule.

A CLASSROOM FOCUS ON QUANTITIES

The seventh-grade students' classroom environment emphasized a focus on quantities and how they were related to one another. The students' reasoning included searching for relationships more than patterns, and their generalizations most often took the form of statements of continuing phenomena. Students searched for relationships connected to speed or gear ratios, because they focused on realistic problems within those two contexts. An example of searching for a relationship can be seen in Dora's work (see **fig. 4**). The table represents the number of rotations that a small gear with 8 teeth (S) and a large gear with 12 teeth (L) made when connected. Unlike the tables that the eighth graders encountered, the values of the gear table did not increase by a uniform amount. Students had to figure out whether every pair in

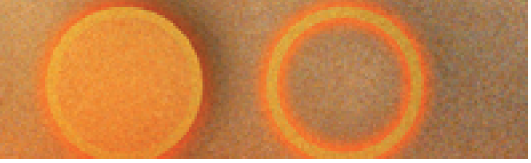


Fig. 4 Dora's table of revolutions for a small gear with 8 teeth and a large gear with 12 teeth

S	L
7 1/2	5
27	18
4 1/2	3
16	10 2/3
1/10	1/15

the table could have come from the same gear pair. Dora focused on the quantities of the gear rotations as she reasoned about the relationship between the gears:

Dora: Think of a gear . . . one gear has 8 teeth, the other has 12. When you spin them, teeth pass through each other. For every 2/3 of the teeth passed on the large one, that's 8 teeth; the small one turns once. If the small one goes 3 turns, the large one will go 2. So if the small goes 7 1/2 times, the large gear will go 5. You can figure this out by setting it up in a fraction . . . 5 over 7.5. Reduce it, and if it equals 2/3, then it's right.

In contrast with Juliana's reasoning, Dora could draw on her understanding of the quantities in the situation to make sense of each pair. The number 2/3 held meaning for her as representing the ratio of the gears' rotations. In addition, Dora's quantitative understanding meant that a justification was embedded in her generalization: Because each tooth on the large gear matches with a tooth on the small gear, 8 teeth (or 2/3 of the teeth) on the large gear will equal one full rotation of the small gear.

The classroom focus on quantities was also reflected in the students' statements of continuing phenomena. An example of this type of statement can be seen in Timothy's work with a speed situation **(fig. 5)**. He worked

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with a $y = mx + b$ table of distance and time values, presented during class, in which a clown began walking a certain number of centimeters away from his home. The students were asked to find out whether the clown walked a constant speed or not by examining the table.

Timothy examined how far the clown walked between 13.25 cm and 17 cm (3.75 cm) and in how much time (6 seconds). After dividing, he found the clown's speed for that section of the journey was 0.625 cm per second. Timothy then checked to see whether that speed was stable throughout the table, which it was. Timothy explained, "For every second, he goes 5/8 of a centimeter." This generalization is a statement of continuing phenomena because it reflects a focus on the dynamic relationship between quantities; it is characterized by a sense of continuation or extension. Students noticed and referenced continuing phenomena statements when they paid attention to the quantities in question. Both speed and gear rotations were characterized by a smooth, dynamic motion.

Fig. 5 Timothy's distance-and-time table

Total Centimeters	Seconds
13.25	10
17	16
22.625	25
32	40
44 1/2	60

CAN FOCUSING ON QUANTITIES HELP STUDENTS LEARN ABOUT LINEARITY?

The generalizations that the eighth-grade students made about linear functions were all similar to Juliana's statement that there must be a pattern "every time," or a table must have stable patterns down both columns. Although this statement is correct for tables that increase by a uniform amount, such as the amounts in **figure 3a**, it will not be the case for nonuniform tables of data, as seen in **figure 4**. When the eighth-grade students did encounter nonuniform $y = mx$ tables or any $y = mx + b$ data, they struggled to make sense of those new problems and declared the data to be nonlinear. Mario explained why he thought a nonuniform table could not represent linear data: "It has to be a pattern that doesn't change. You know? It has to be like 3, 6, 9, like that."

In contrast, the seventh-grade students were able to extend their understanding of speed and gear ratios to $y = mx + b$ situations. Their focus on the quantities helped them develop an appropriate understanding of what has to happen for data to be linear:

Larissa: Because the relationship is staying the same throughout the whole table. That's kind of a general rule.
Teacher: What's the general rule?
Larissa: If the relationship stays the same the whole time, if the frog walks the same pace the whole time, or if you spin the gears the same . . . it will be a line.

Larissa and her classmates could explain and justify their solutions and generalizations, as seen in Dora's work. However, the eighth-grade students struggled to provide coherent explanations, as seen with Juliana's work. For instance, Larissa was able to explain how she made sense of the data in **figure 5**:

Larissa: Because you're finding out how many seconds it takes him to go x amount of cm or how many cm you go in x amount of seconds. So if he's going the same, if it's taking him the same time to go the same amount of cm, the whole, throughout the whole table, he's going the same speed.

Timothy could then connect Larissa's idea of speed to the idea of the slope of a graph, exclaiming, "I've been saying that the pace is the slope!" His explanation involved using numbers from a similar problem with a speed of 4/5 of a second per centimeter:

Timothy: The slope means that whatever x goes up by, 4/5 of that is how much y goes up by.
Teacher: And what does 4/5 have to do with the speed?
Timothy: It's going basically 4/5 of a second per cm . . . basically it's gonna be continuously the same

fraction for the slope. And basically that means it's gonna be the same units per whatever, per whatever amount of time.

Notice that the students' generalizations about linearity relied on references to the quantities of centimeter, second, and speed. The classroom emphasis on these quantities, and the ways in which the students reasoned with them, suggests that working with quantitative relationships can support students' abilities to extend and justify their generalizations.

Although the eighth-grade students could at times make correct generalizations, these generalizations were not well connected to other knowledge. The students struggled to extend their ideas to new situations, such as $y = mx + b$ scenarios. The seventh-grade students, however, were able to explain and support their ideas specifically by appealing to their understanding of quantities in the situation. Their tendency to rely on a quantitative understanding to support their reasoning suggests that helping students make correct generalizations is not the only goal to keep in mind; in addition, we should try to help students generalize from relationships that are meaningful to them.

HOW TEACHERS CAN HELP STUDENTS FOCUS ON QUANTITIES

Teachers play an important role in helping students focus on quantities and relationships. One way to foster these ideas when introducing linear functions is to develop the idea of linearity as an experiential quantity, such as gear ratio or speed. Problems like the Connected Gears problem and the Frog Walking problem are appropriate for helping middle school students coordinate changes between quantities and connect linear growth to a phenomenon such as constant

Fig. 6 Additional problems to check for understanding

Gear Follow-Up Problems

1. If you could replace gear A with a new gear that would make gear B turn twice as fast, how many teeth would the new gear have?

2. You want to replace gear A with a different gear to make gear B turn twice as slow instead of twice as fast. How many teeth would that different gear have?

3. Right now, gear A has 8 teeth and gear B has 12 teeth. Can you think of two different sizes for gear A and gear B so that the gear ratio would still be the same? How many possibilities can you find?

4. Gear C has 6 teeth. You hook it up to gear D and turn it a certain number of times. Gear D turned 1/4 of the number of turns that gear C turned. How many teeth would gear D have to have?

5. Gear E has 15 teeth. If you turn it 12 times, gear F turns 10 times. How many teeth does gear F have to have?

6. Ricardo was working with gear A and gear B (8 teeth and 12 teeth). He turned the gears a certain number of times. He is not sure exactly how many times, because he lost count. Then he turned gear A one extra rotation. How many extra rotations did gear B turn?

We should try to help students generalize from relationships that are meaningful to them

speed. Once students have had the opportunity to experiment with rotating different types of gears or timing each other while trying to walk at a constant rate, pose some additional problems to explore relationships more deeply. The follow-up problems in **figure 6** are designed to encourage students to create ratios and connect the idea of constant ratio to a linear function.

Number pattern problems that introduce only the first few terms, either in tabular form or another form, are not mathematically well defined. There is more than one function, for instance, that could have 5, 12, and 19 for the first three terms. Teachers may assume that students will know that they should develop the appropriate linear function. However, there are many instances in which students find different patterns, some of which are more algebraically useful than others (English and Warren 1995). The Connected Gears problem examined here, for example, can help both students and teachers focus on the same rule that is tied to the context of the problem.

Students may want to focus only on number patterns once they have developed tables or formulas. Intervene to draw students' attention back to the quantities at hand. To do so, incorporate the language of quantities into the classroom discussion. When students describe patterns such as "each time x goes up by 5, y goes up by 2," a teacher could ask students in this context to explain whether this statement means that the clown walked the same distance throughout or whether he or she sped up or slowed down. In the case of gears, a teacher could ask students to explain whether this pattern means that each pair in a number table came from the same two gears or whether the pairs could have come from different gears.

It is important to be careful, however, when selecting situations and problems. Some linear function problems use contrived contexts in which data would not realistically be linear, such as the relationship between the number of surf boards sold and the temperature at the beach. The unrealistic nature of these problems could conflict with students' natural sense-making ability about linearity. Similarly, presenting problems with data that are only approximately linear, either because of measurement error or the inexact nature of the phenomenon, could prevent students from building appropriate relationships that isolate the importance of ratios.

Although inexact data can provide valuable problem contexts, especially in terms of highlighting the power of mathematics for making sense of messy situations, these contexts are best reserved for students who have already formed an understanding of linearity as being a constant rate of change.

A teacher's role is critical in terms of choosing the right problems, shaping a classroom discussion, posing appropriate questions, and guiding students to think carefully about quantities. The seventh-grade students from this study demonstrated that they could develop very powerful ideas about linearity if allowed to support their reasoning with quantities. Their success suggests that if we encourage students to build ratios from relationships between quantities, they will be poised to make correct, well-supported generalizations about linear functions.

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