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# Algebraic Thinking in the Elementary and Middle Grades

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Two decades ago, Schoenfeld (1995) wrote the following:

> Algebra has become an academic passport for passage into virtually every avenue of the job market and every street of schooling. With too few exceptions, students who do not study algebra are therefore relegated to menial jobs and are unable often to even undertake training programs for jobs in which they might be interested. They are sorted out of the opportunities to become productive citizens in our society. (pp. 11–12)

The inequity captured in these words conveyed, at the time, an increasing awareness of algebra's "gatekeeper" effect in the United States, where high failure rates in school algebra kept large numbers of students from career and economic opportunities, particularly in fields involving science, technology, engineering, and mathematics (Kaput, 2008; Moses & Cobb, 2001; RAND Mathematics Study Panel, 2003; Stigler, Gonzales, Kawanaka, Knoll, & Serrano, 1999; U.S. Department of Education, 1997, 1998, 1999). Such challenges are not unique to the United States, and similar difficulties have been described in other countries (e.g., Cooper & Warren, 2011; Herscovics & Linchevski, 1994; Subramaniam & Banerjee, 2011).

The historical "arithmetic-then-algebra" approach to school mathematics left students with little cognitive space to negotiate the abrupt transition from years of computational work in the elementary and middle grades to the abstract concepts of formal high school algebra.[1] Recognizing this, scholars formulated new recommendations for teaching and learning algebra. Many notable conferences and working groups convened during this period, including the 7th International Congress on Mathematical Education (1992); the U.S. Department of Education Algebra Initiative Colloquium (1993); the Nature and Role of Algebra in the K–14 Curriculum Conference (1998), held jointly by the National Council of Teachers of Mathematics (NCTM) and the Mathematical Sciences Education Board; the 12th International Commission on Mathematical Instruction (ICMI) Study Conference on the Future of the Teaching and Learning of Algebra (2001); and the Mathematics Learning Committee of the National Research Council. Through them an argument emerged that school algebra should be reformulated as a kindergarten–grade 12 strand of thinking. Since then, this longitudinal approach has become widely instantiated in current reform initiatives in the United States (e.g., NCTM, 2000, 2006; National Governors Association Center for Best Practices [NGA] & Council of Chief State School Officers [CCSSO], 2010; National Research Council [NRC], 2001) and is increasingly represented in K–12 curricular resources.

The research reported here was supported in part by the National Science Foundation under DRK-12 Awards No. 1219605, 1219606, 1415509, and 1154355 and REC Awards No. 0529502 and 0952415. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.



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The notion that algebra instruction might begin as early as kindergarten prompted vigorous ideological debates, influenced by a broader international discourse on teaching and learning school algebra, regarding what "algebra" is, what it means to "do" algebra, and what the forms of thinking that might be considered "algebraic" would resemble, particularly in the largely uncharted area of elementary grades (e.g., Arcavi, 1994; Bell, 1996b; Lacampagne, Blair, & Kaput, 1995). The body of K–grade 8 algebra research that emerged in response to these debates reflected a shift from a focus on students' errors and misconceptions to an emphasis on the meanings students make of algebraic concepts (Bednarz, Kieran, & Lee, 1996; Kaput, 2008; Kieran, 2007).

The narrative that developed, particularly among researchers who studied algebraic thinking in the lower grades, reflected a deeply held view that students can begin to reason algebraically much earlier than previously thought and in ways that can potentially ameliorate the difficulties students have historically faced in high school algebra. Importantly, this perspective did not espouse teaching children in elementary and middle grades the formal procedural algebra of traditional ninth-grade algebra 1 courses. *Early algebra*—that is, algebraic thinking in elementary and middle grades—should not be conflated with these courses or with the prealgebra courses that are common in U.S. middle schools. Indeed, early algebra is *not* "algebra early" (Carraher, Schliemann, & Schwartz, 2008). Rather, early algebra gives students opportunities to engage in age-appropriate forms of algebraic reasoning that in many cases build from students' everyday experiences. As Malara (2003) argued, the cognitive framework for thinking algebraically should begin in the earliest years of schooling so that children can understand arithmetic from an algebraic perspective.

We use this historical background as a point of departure for our chapter's focus on the algebraic thinking of students in kindergarten through grade 8. In this, we acknowledge that some differences still exist regarding what constitutes algebra or algebraic thinking and that these differences have implications for teaching algebra, particularly in the elementary grades. (For example, there are differing views on the role of variable notation in elementary grades mathematics.) Our goal here is not to revisit these arguments. (For a careful and comprehensive treatment of the historical perspectives on teaching and learning algebra and algebra research, we refer the reader to Carraher and Schliemann, 2007, and Kieran, 2007.) Instead, we take Kaput's (2008) analysis of algebra and algebraic thinking—one of the frameworks produced during this earlier period that is well regarded for its analysis of algebra content—and use it as an organizing lens for reviewing K–grade 8 algebra research in more recent years, focusing primarily on the last decade. First, we briefly summarize Kaput's framework and describe its role in this chapter.

### *Kaput's Core Aspects as an Organizing Framework*

Kaput (2008) argued that algebraic thinking is composed of two core aspects: (1) generalizing and representing generalizations in increasingly conventional symbolic systems and (2) syntactically guided reasoning and actions on generalizations represented in conventional symbolic systems. Stated another way, algebraic thinking might be viewed as the four core practices of *generalizing, representing, justifying,* and *reasoning with* mathematical structure and relationships (Blanton, Levi, Crites, & Dougherty, 2011).

While each of these practices will be thoroughly discussed in this chapter, there are a few points we wish to make here. First, generalizing is widely viewed as the heart of algebraic thinking and a core mathematical activity (Bell, 1996a; Cooper & Warren, 2011; Kaput, 2008; Kieran, 2007; Mason, 1996; Radford, 2006). Definitions of generalization have varied throughout the history of research in mathematics education, with earlier views situating generalization as an individual, cognitive construct (e.g., Kaput, 1999). More recent sociocultural definitions have positioned generalization within activity and context, describing generalization as a collective act, distributed across multiple agents (Lobato, Ellis, & Muñoz, 2003; Reid, 2002; Tuomi-Gröhn & Engeström, 2003). Researchers with this perspective attend to how social interactions, tools, and history shape people's generalizing activity, viewing generalization as a social practice rooted in activity and discourse (Jurow, 2004; Latour, 1987). Both perspectives are important because of the dual focus on individual students' mechanisms for generalization and the instructional conditions that support students' generalizing. Thus, we borrow from both the cognitive and sociocultural traditions to define generalizing (see also Ellis, 2011b) as a construct in which learners in specific sociocultural contexts engage in activity that can be framed in one of the following ways: (*a*) identifying commonality across cases (Dreyfus, 1991), (*b*) extending one's reasoning beyond the range in which it originated (Carraher, Martinez, & Schliemann, 2008; Harel & Tall, 1991; Radford, 2006), or (*c*) deriving broader results from particular cases

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(Kaput, 1999). We use the term *generalizing* to refer to any of these processes, whereas *generalization* refers to the outcome of these actions.

Second, representing or symbolizing generalizations—an act by which a multiplicity is compressed into a single unitary, "generalized" form (Kaput, Blanton, & Moreno, 2008)—is arguably of equal importance. The generalizations students notice remain hidden without symbolic systems to represent them and provide objects with which students might reason. As Kaput et al. (2008) described, generalizing and symbolizing are tightly linked in that symbols allow generalizations to be expressed in a stable and compact form. The symbolization process begins with an intra- or extramathematical classroom situation (A) from which a written, oral, or drawn description of the situation (B) is built and tested against an observation of the original experience. The resulting (new) symbolization ($A_B$) of the student's experience is then refined. This socially mediated process (see also Malara, 2003; Meira, 1996; Radford, 2000) is repeated until a conventional and compact symbolization (for the particular classroom) is reached.

The symbols themselves can also be acted on and manipulated as objects in their own right, without concern for their referents, as a process of reasoning with generalized forms. Although there is some debate about which symbolic systems might be viewed as algebraic, we interpret such systems here broadly to include not only variable notation, but also other symbolization systems such as natural language, coordinate graphs, and tables (Carraher & Schliemann, 2007; Kaput, 2008). Indeed, rather than privileging one single representation, scholars have argued (e.g., Brizuela & Earnest, 2008; Duval, 2006) that students should be able to coordinate different representations of the same object and shift flexibly among them. Finally, although the algebraic nature of generalizing is apparent, the feature that elevates the three practices of representing, justifying, and reasoning as used here to forms of *algebraic* thinking is when they are performed in the service of *actions with or on generalizations.*

Although there has been some difficulty in uniformly parsing algebra from a content perspective (Carraher & Schliemann, 2007)—a difficulty Kaput (2008) also acknowledged—Kaput identified several content strands in which his core aspects occur (see Table 15.1 for a description of Kaput's core aspects and strands). Two of these strands (Strands 1 and 2) reflect content around which much of the research on algebraic thinking in K–grade 8 has matured: generalized arithmetic and quantitative reasoning (Strand 1) and functional thinking (Strand 2). Thus, in this chapter, we use Kaput's core aspects and strands as a pragmatic framework for presenting recent research on K–grade 8 students' activity of generalizing, representing, justifying, and reasoning with mathematical structure and relationships within Strands 1 and 2. We therefore organize the chapter into the following three sections to reflect these content areas: Generalized Arithmetic, Functional Thinking, and Quantitative Reasoning. For each of these content areas, we first briefly identify some of its foundational ideas in relation to algebraic thinking and then consider recent research within these areas around the core practices.

**TABLE 15.1.** Kaput's Core Aspects and Strands

| The Two Core Aspects |
|---|
| (A) Algebra as systematically symbolizing generalizations of regularities and constraints. |
| (B) Algebra as syntactically guided reasoning and actions on generalizations expressed in conventional symbol systems. |
| **Core Aspects A & B Are Embodied in Three Strands** |
| 1. Algebra as the study of structures and systems abstracted from computations and relations, including those arising in arithmetic (algebra as generalized arithmetic) and in quantitative reasoning. |
| 2. Algebra as the study of functions, relations, and joint variation. |
| 3. Algebra as the application of a cluster of modeling languages both inside and outside of mathematics. |

*Note.* From "What Is Algebra? What Is Algebraic Reasoning?," by J. J. Kaput, in J. J. Kaput, D. W. Carraher, and M. Blanton (Eds.), *Algebra in the Early Grades* (p. 11), 2008, Mahwah, NJ: Lawrence Erlbaum/Taylor & Francis Group; Reston, VA: National Council of Teachers of Mathematics.

## Generalized Arithmetic

Particularly in the elementary grades, students traditionally spend much of their time in mathematics classrooms performing arithmetic computations and producing "answers." Generalized arithmetic involves a larger purpose within the arithmetic of numbers: students look across multiple computations; notice and represent underlying structure, such as fundamental properties of operations (e.g., field axioms such as the commutative property of addition), or relationships in operations on classes of numbers; and justify and reason with the generalizations observed (Kaput, 2008). Generalized arithmetic also includes students developing an understanding of the fundamental con-

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cept of equivalence and reasoning with representations of equivalence (or lack thereof) in the form of equations and inequalities (Kaput, 2008). Because this work naturally builds on the arithmetic work with which students and teachers, particularly in elementary grades, are already quite familiar, a number of researchers (e.g., Carpenter, Franke, & Levi, 2003; Davis, 1985; Kaput, 2008; Russell, Schifter, & Bastable, 2011a) have argued that such activity can serve as the basis for introducing students to algebraic thinking.

### *Generalizing and Reasoning With Arithmetic Relationships*

Generalizing and reasoning with arithmetic relationships and structure includes noticing regularity in arithmetic operations that can be generalized beyond the given cases and making use of these generalizations to solve problems. It includes generalizing fundamental properties of operations, explicitly identifying properties underlying computational strategies, and developing generalizations about special classes of numbers. All these contexts, however, rely on students' understanding of the equals sign as an indication of equivalence. We start, then, by examining research on students' understanding of the equals sign, then address how students generalize and reason with arithmetic relationships.

**Mathematical equivalence as a foundation for generalizing arithmetic.** A fundamental component of children's mathematical work is their understanding of the equals sign and its role in representing equivalent expressions, whether in computational work (e.g., $3 + 7 = \_\_\_$), in equations with a fixed unknown (e.g., $12 = 3 + x$), or in generalized patterns such as fundamental properties of operations. Students' misconceptions about the meaning of the equals sign have been well documented and have been characterized by an interpretation of the equals sign as a symbol indicating the need to compute an answer (commonly referred to as an "operational" view of the equals sign) as opposed to a symbol indicating the mathematical equivalence of two quantities or expressions (commonly referred to as a "relational" view of the equals sign; e.g., Carpenter et al., 2003; Knuth, Alibali, McNeil, Weinberg, & Stephens, 2005; Knuth, Stephens, McNeil, & Alibali, 2006; Molina & Ambrose, 2008). Along these lines, a variety of tasks have been used to diagnose students' conceptions of the equals sign, including equation-solving items such as $8 + 4 = \square + 5$ (Carpenter et al., 2003), equation-structure items such as deciding if $57 + 22 = 58 + 21$ is true or false (Blanton, Stephens, et al., 2015), and equivalent equations items such as determining if the equations $2 \times \square + 15 = 31$ and $2 \times \square + 15 - 9 = 31 - 9$ have the same solution (Knuth et al., 2005). Researchers have also directly asked students to define the equals sign (Knuth et al., 2006) or evaluate the quality of definitions such as "the total," "the end of the problem," and "that two amounts are the same" (McNeil & Alibali, 2005).

Several researchers have noted connections between students' understanding of the equals sign and their experiences solving arithmetic problems. McNeil, Fyfe, Petersen, Dunwiddie, and Brletic-Shipley (2011) found that even without explicit instruction about the equals sign, 7- and 8-year-old students who completed arithmetic problems presented in nontraditional formats (e.g., $\_\_ = 9 + 4$) in written practice sessions later demonstrated a better understanding of equivalence—on a written assessment involving equation solving, equation encoding, and defining the equals sign—than did those who completed items with traditional formats or had no practice session. Using the theory of "change resistance" (McNeil & Alibali, 2005), McNeil et al. (2011) argued that students' operational view of the equals sign becomes entrenched after seeing problems in exclusively standard formats throughout the early grades and that opportunities to complete nontraditional items weaken this entrenchment and expose children to patterns that facilitate acquisition of a relational view of the equals sign. A study by DeCaro and Rittle-Johnson (2012) supports the utility of allowing children to explore equals sign problems on their own prior to instruction. They found that children in grades 2–4 who were asked to solve relatively unfamiliar problems (e.g., $3 + 5 = 4 + \square$) prior to receiving brief instruction that included nonstandard equations and a relational explanation of the equals sign exhibited significantly higher conceptual knowledge compared to students who experienced the same instruction through a conventional instruct-then-practice approach on both an immediate written posttest and a delayed retention test.

Other researchers have found that students' understanding of the equals sign matters in terms of their performance on traditional algebra tasks as well. Alibali, Knuth, Hattikudur, McNeil, and Stephens (2007) found that a more sophisticated understanding of the equals sign was associated with better performance on equivalent equations problems in grades 6–8. In addition, students' performance on these problems varied as a function of *when* they acquired a sophisticated

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understanding of the equals sign. Those who acquired a relational understanding of the equals sign closer to the beginning of sixth grade exhibited more success in solving equivalent equations problems on a written assessment at the end of eighth grade. Knuth et al. (2006) similarly found that middle school students' understanding of the equals sign is associated with their performance solving simple linear equations. Even when controlling for mathematics ability as measured by standardized achievement test scores, those who held an operational view of the equals sign had more difficulty solving equations than those who held a relational view. Matthews, Rittle-Johnson, McEldoon, and Taylor (2012) likewise found that students in the elementary grades with a relational understanding of the equals sign were more successful at solving simple algebraic equations such as $c + c + 4 = 16$ on a written assessment. This all points to a central theme: a relational understanding of the equals sign, coupled with a familiarity with variable notation, supports students in solving simple linear equations by allowing them to focus on the meaning of the equals sign within an equation rather than simply applying a sequence of memorized procedures (e.g., Blanton, Stephens, et al., 2015; Carpenter et al., 2003; Matthews et al., 2012).

Recent research has further fleshed out the operational/relational dichotomy to provide a more nuanced perspective on students' understanding of mathematical equivalence and the meaning of the equals sign, with some noting distinctions such as operational, relational-computational, and relational-structural in students' thinking (Stephens et al., 2013). Along these lines, Matthews et al. (2012) and Rittle-Johnson, Matthews, Taylor, and McEldoon (2011) designed a written assessment of equivalence understanding that placed previously incommensurable items onto a single scale to compare item difficulty for children at different ability levels. They assessed grades 2–6 students' knowledge of mathematical equivalence using a variety of item classes (e.g., equation solving, equation structure, requests for definition) and equation structures (e.g., equations of the form $a + b = c$, $a = a$, $c = a + b$, $a + b = c + d$). They found that the structure of the equation had a large influence on performance (equations with operations on both sides of the equals sign proving most difficult), although item class did not. Generating a relational definition of the equals sign was especially difficult for students, even more so than solving or evaluating equations with operations on both sides. Recognizing a relational definition from a list or rating a relational definition as a good definition, on the other hand, was much easier for students than producing one. Overall, Matthews et al. (2012) and Rittle-Johnson et al. (2011) found great variability among children who might respond similarly to any one particular item. For example, students who were unsuccessful in providing a relational definition of the equals sign varied a great deal in the degree to which they were successful in determining whether the equation $4 = 4 + 0$ was true or false. The findings of Matthews et al. (2012) and Rittle-Johnson et al. (2011) led to the development of a four-level construct map describing a progression of understanding from rigid operational to comparative relational views (see Table 15.2). T2

Jones and Pratt (2012; see also Jones, Inglis, Gilmore, & Dowens, 2012) suggested that the notion of substitution of an expression by an equivalent one, which Kaput (2008) characterizes as a syntactic aspect of algebra from the structure of arithmetic, is also an important part of a sophisticated understanding of mathematical equivalence. They proposed that a complete understanding of the equals sign involves both sameness and substitutive components. The sameness component entails holding a "relational" view of the equals sign as defined above—that is, understanding that the symbol indicates the equivalence of two mathematical objects. The substitutive component, on the other hand, entails understanding that any number or expression can be replaced by an equivalent one. For example, in the expression $30 + 41$, 41 can be replaced by $40 + 1$ if we know $41 = 40 + 1$ (Jones & Pratt, 2012).

Jones and Pratt (2012) argued that substitution is important for understanding equivalence in algebraic contexts. In a cross-cultural study involving 11- and 12-year-old English and Chinese students, Jones et al. (2012) found that a substitutive conception is a way of understanding the equals sign that is distinct from the sameness conception. They reported no consistent order in which the two conceptions develop, but found that children with a sophisticated conception of the equals sign explicitly endorsed the substitutive conception when asked to rate the "cleverness" of different definitions of the equals sign on a written assessment.

Hattikudur and Alibali (2010) suggested that students' understandings of the equals sign can be enhanced when learning about this symbol takes place alongside learning about inequality symbols. They assigned third- and fourth-grade students to a "comparing symbols" group, "equals sign" group, or control group and had them complete a written assessment prior to and at the conclusion of a brief lesson. Hattikudur and Alibali found that students in the "comparing symbols" group showed greater gains in developing a relational understanding of the equals sign than did students who received instruc-

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**TABLE 15.2.** Construct Map for Knowledge of the Equal Sign as Indicator of Mathematical Equality

| Level | Description | Core equation structure(s) |
|---|---|---|
| Level 4: Comparative Relational | Successfully solve and evaluate equations by comparing the expressions on the two sides of the equal sign, including using compensatory strategies and recognizing transformations maintain equality. Consistently generate a relational interpretation of the equal sign. | Equations that can be most efficiently solved by applying simplifying transformations: For example, without adding 67 + 86, can you tell if the number sentence "67 + 86 = 68 + 85" is true or false? |
| Level 3: Basic Relational | Successfully solve, evaluate, and encode equation structures with operations on both sides of the equal sign. Recnogize relational definition of the equal sign as correct. | Operations on both sides:<br>$a + b = c + d$<br>$a + b - c = d + e$ |
| Level 2: Flexible Operational | Successfully solve, evaluate, and encode atypical equation structures that remain compatible with an operational view of the equal sign. | Operations on right:<br>$c = a + b$<br>No operations: $a = a$ |
| Level 1: Rigid Operational | Only successful with equations with an operations-equals-answer structure, including solving, evaluating, and encoding equations with this structure. Define the equal sign operationally. | Operations on left:<br>$a + b = c$ (including when blank is before the equal sign) |

*Note.* From "Measure for Measure: What Combining Diverse Measures Reveals About Children's Understanding of the Equal Sign as an Indicator of Mathematical Equality," By P. G. Matthews, B. Rittle-Johnson, K. McEldoon, and R. S. Taylor, 2012, *Journal for Research in Mathematics Education, 43*(3), p. 320.

tion focused only on the equals sign. Moreover, students in the "comparing symbols" group scored higher on post-test items that assessed knowledge about inequality symbols and inequality problem solving. These findings support the approach taken by Dougherty's (2008) Measure Up project (see the Quantitative Reasoning section in this chapter) in which students learned to compare and describe measurements in terms of equal to, not equal to, greater than, and less than.

Finally, there exists some consensus among researchers that developmental readiness is not the issue driving students' tendencies to hold operational as opposed to relational conceptions of the equals sign (e.g., Carpenter et al., 2003; McNeil et al., 2011), but rather that these tendencies are a reflection of how these concepts are addressed—or not—in instruction. For instance, analyses of textbooks in the elementary grades (Powell, 2012; Seo & Ginsburg, 2003) and middle grades (McNeil et al., 2006) revealed little if any explicit discussion of the meaning of the equals sign or inclusion of nonstandard equation types (i.e., other than $a + b = c$, for example), despite the fact that instruction about nonstandard equations has been shown to promote relational understanding of the equals sign (e.g., Blanton, Stephens, et al., 2015; Falkner, Levi, & Carpenter, 1999; McNeil & Alibali, 2005; McNeil et al., 2006; Molina & Ambrose, 2008; Powell & Fuchs, 2010). This points to considerable challenges for curriculum and instruction to be more responsive to the significant progress made over the last decade-plus in identifying real pathways for building children's relational understanding of the equals sign.

**Generalizing and reasoning with relationships underlying operations.** Students in the elementary grades are capable of generalizing and reasoning with arithmetic relationships while solving problems that at first glance might involve nothing more than computation. Fundamental properties of operations, in particular, provide valuable opportunities for students in the elementary grades to generalize and reason with the mathematical relationships that they notice in computational work and that already make sense to them. In this regard, Carpenter and colleagues (Carpenter et al., 2003; Carpenter & Levi, 2000) found true/false and open number sentences to be fruitful contexts for eliciting students' generalizations about such properties. Discussions exploring whether arithmetic equations such as 58 + 0 = 58 and 789,564 + 0 = 789,564 were true encouraged students in a second-grade classroom to articulate the generalization, "Zero added with another number equals that other number" (Carpenter et al., 2003). Blanton, Stephens, et al. (2015) likewise found that third-grade students who experienced early algebra instruction involving a broad range of algebraic ideas (including algebra as generalized arithmetic) showed significant gains in their ability to recognize and state fundamental properties, such as the commutative property of addition, through computational work.

There are other opportunities for children to notice regularities in how operations behave. Schifter, Bastable, Russell, Seyferth, and Riddle (2008) reported on a kindergarten classroom where students played the game "Double Compare," in which pairs of students select two cards apiece, each bearing a numeral from 1 to 6, and the player with the higher combined total wins. Although the goal of the task might be perceived as providing computational practice, students reasoned algebraically by implicitly working with the generalization that if two of the numbers being compared were the same, they could be "ignored" and only the other numbers need be compared. Applying this reasoning to the pairs 6 and 2 and 6 and 1, one student argued that because "these are the same [the sixes], this [the 6 and 2] must be more" (p. 265). Students made this generalization explicit when the teacher asked, "Does this only work for 6?" by noting that any numbers that are the same could be disregarded. Here, students reasoned with the generalization that if one number is greater than another, and the same number is added to each, the first total will be greater than the second. Schifter et al. (2008) argued that providing students opportunities to reason with such regularities and examine structure in the number system would ease the transition to more formal approaches in their future work.

Similarly, Russell et al. (2011a) reported on students in a first-grade classroom examining ways to decompose 10. Although such a task develops operation sense and the idea of equivalence, it also provides students opportunities to generalize relationships about how operations behave. Once students had generated a list of ways to make ten—5 + 5, 4 + 6, and so on—the teacher asked students why they were equivalent. One student explained, "if you start out with 5 plus 5, then you . . . and then take 1 away from this 5 and add it to this 6, and then this is 4 and so on and so on" (p. 62), verbalizing a precursor of the more formal generalization, "Given two addends, if 1 is subtracted from one addend and 1 is added to the other, the sum remains the same" (p. 62).

The concept of compensation illustrated here not only is an important attribute of generalized arithmetic (Kaput, 2008) but has also been found to come quite naturally to children even before their work with specific numbers (see also the approach used by Dougherty and colleagues in the Quantitative Reasoning section in this chapter). In a study of children's understanding of operations before being introduced to arithmetic, Britt and Irwin (2011) conducted interviews with children in which they asked what would happen to a total collection of candy divided into two boxes, without specifying the number of candies in each box, under the following scenarios: (1) a piece was removed from one of the boxes; (2) a piece was added to one of the boxes; (3) a piece was moved from one box to the other; (4) a piece was removed from one box and the interviewer added a different piece to the other box. The 4-year-old children were certain that the total quantity of candies would stay the same if a piece of candy was moved from one box to another or replaced with a different piece but would increase or decrease if the amount in just one of the boxes was altered. The 5- and 6-year-old children could additionally explain relationships in such a way that showed they understood generalities underlying the movement of candies. However, given a similar task with numbers, such as asking if 5 + 5 would be the same as 4 + 6, most students were unsuccessful until age 7. The authors concluded that instruction should build on the nascent generalizations children bring to formal schooling (cf. Mason, 2008) so that the complexity of learning to understand numbers does not distract them from using this preexisting knowledge.

Similarly, in a study with children older than those involved in Britt and Irwin's (2011) study, Schliemann, Lins Lessa, Brito Lima, and Siqueira (reported in Schliemann, Carraher, & Brizuela, 2007) examined 7- to 11-year-old students' understandings of the preservation of equivalence across different physical models and contexts. The researchers interviewed 120 Brazilian students to understand their thinking about equivalence relationships between the following: (*a*) weights in a two-pan balance scale, (*b*) quantities of discrete concrete objects, (*c*) quantities described in word problems, and (*d*) the two sides of written equations. The problems presented included items where all numerical values under comparison were displayed, followed by items that included only partial numerical information, and finally by items in which no numerical values were given. Word problems elicited more attention to transformations and problem structure than did problems with concrete counters, objects on a scale, or equations. Problems presented in these latter contexts were more likely to elicit computation. Items in which all numerical values were known likewise overwhelmingly elicited computation, although those in which the values to be compared were unknown, or partially known, more often encouraged students to focus on problem structure and transformations.

During the elementary grades, much instructional time in mathematics is devoted to developing fluency with multiplication, including learning multiplication facts. Not only are mathematical understanding and computational fluency intertwined (NRC, 2001), alge-

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braic thinking can play an important role in developing both. Carpenter, Levi, Berman, and Pligge (2005) found that third-grade students with at least an implicit understanding of the distributive property were able to use this property to find solutions to multiplication facts they did not yet know to the point of recall. Baek (2008) likewise found that students in grades 3–5 who had an understanding of properties of operations—in particular, the associative property of multiplication and the distributive property—experienced success solving multiplication story problems involving multidigit numbers. Empson, Levi, and Carpenter (2011) extended this work in an important direction, finding that elementary students naturally use properties of operations and equality in their strategies for problems involving operations with fractions. Consider their account of Jill, a fifth grader faced with the following classroom task: "It takes $\frac{3}{8}$ of a cup of sugar to make a batch of cookies. I have $5\frac{1}{2}$ cups of sugar. How many batches of cookies can I make?"

> She said she knew that 8 three-eighths would be 3, which means that 4 three-eighths would be half that much, or $1\frac{1}{2}$, and 12 three-eighths would therefore be $4\frac{1}{2}$. At this point, she knew that she needed only 1 more cup to use up all $5\frac{1}{2}$ cups. Again Jill used the relationship between $\frac{3}{8}$ and 3 as a reference point. She said that because 8 three-eighths was 3, a third as many three-eighths would be a third as much, or 1. That is, $(\frac{1}{3} \times 8) \times \frac{3}{8}$ is 1, and $\frac{1}{3} \times 8$ is $\frac{8}{3}$ or $2\frac{2}{3}$. She concluded that she could make a total of $12 + \frac{8}{3}$ batches, which would be equal to $14\frac{2}{3}$ batches. (p. 418)

In unpacking Jill's strategy, Empson et al. (2011) detailed the implicit use of several properties, including the additive property of equality, the multiplicative property of equality, the associative property of multiplication, and the distributive property. Although the students described by Carpenter et al. (2005), Baek (2008), and Empson et al. (2011) did not explicitly identify these properties in use—thus, arguably, their actions are not yet *algebraic* (Kaput, 2008)—these researchers argued that students who are able to intuitively reason with generalized properties rather than relying on computational procedures alone are better prepared for formal algebra.

Although research on students' reasoning with properties is very promising, Bastable and Schifter (2008) caution against attributing to students generalizations that might not yet be present in their thinking. As students implicitly use properties of operations, their underlying thinking is arguably more arithmetic than algebraic in nature if their focus remains on specific values rather than generalized structures. However, in spite of the care to be taken around the sometimes blurred boundaries between arithmetic and algebraic thinking, there is strong evidence that young students are capable of reasoning with properties of operations in their computational work, reasoning which is productive in their preparation for algebra.

**Generalizing and reasoning with special classes of numbers.** Researchers have found that elementary students can also generalize and reason with arithmetic relationships about special classes of numbers. Bastable and Schifter (2008) described a scenario from a first-grade classroom in which "snowmen," represented by beans painted as snowmen, were invited to a "Snow Ball," but they could only come as partners. As students made observations over the course of several days about what numbers of snowmen could or could not go to the ball, they came up with different observations, such as "Each time you add one to a group that can go, you get a group that can't" (p. 176). Although not yet using the terms "even" and "odd," students could generalize and reason with relationships about classes of numbers through informal instantiations of more formal generalizations (e.g., "an even number plus one equals an odd number").

Elementary students' generalizations are not limited to those regarding operations on even and odd numbers. Researchers have found that elementary students can generalize relationships about other classes of numbers, including square numbers (Bastable & Schifter, 2008), consecutive square numbers (Bastable & Schifter, 2008), and numbers involving exponents (Lampert, 1990). Students have also made generalizations about factors and divisibility rules (Carpenter et al., 2003). Such findings support the argument that arithmetic is a fruitful context for generalizing and reasoning with relationships from the very start of formal schooling. And as we explore next, the generalizations students notice can provide rich opportunities for representing and justifying arithmetic relationships.

### *Representing Arithmetic Relationships*

The representations students use for arithmetic relationships can take many forms. However, because of the important role variable plays as an artifact of formal algebra (Kline, 1972), we focus throughout this chapter on research concerning the role of variable notation in

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students' thinking. Although students at the secondary level have faced many difficulties working with and interpreting variable notation (MacGregor & Stacey, 1997), there is evidence that elementary-grade children in supportive classroom environments are capable of using variable notation in sophisticated ways (Brizuela & Earnest, 2008). Drawing parallels between learning language and learning variable notation, Brizuela and Earnest (2008) argue that just as we do not shy away from introducing children to the complexities of natural language, we should not avoid introducing variable notation to younger students.

Variables typically take on different roles in students' K–8 experiences, including variable as a varying quantity in algebraic expressions (see the Functional Thinking section in this chapter), variable as a fixed unknown in an equation, or variable as a generalized number in relationships that represent properties of operations (Blanton et al., 2011). The generalizations that students make in the realm of generalized arithmetic—whether about properties of operations or other arithmetic generalizations such as the structure underlying compensation strategies—can serve as a context for developing students' abilities to represent mathematical ideas symbolically. Variables in this case serve the role of generalized number. The symbolized generalizations can in turn become objects of study that allow for new insights to be made (Kaput et al., 2008), supporting Brizuela and Earnest's (2008) argument that introducing children to mathematical notation enhances their understanding of content.

Fujii and Stephens's (2008) study suggested that even before students interpret or use literal symbols as variable notation, they are capable of engaging in what Fujii and Stephens have termed "quasi-variable thinking." This thinking is evident when students offer general explanations of why number sentences like $78 - 49 + 49 = 78$ are true. Although students may not yet grasp the full range of variation implied by $78 - a + a = 78$, equations like $78 - 49 + 49 = 78$ can serve as a bridge to more formal variable notation. Fujii and Stephens (2008) found that second- and third-grade students spontaneously used their own "variables" (e.g., triangle, circle, square) to represent a range of numbers when reasoning about equations such as these.

Carpenter et al. (2003) argued that when the ideas to be represented are ones that students already understand (e.g., properties of operations that they themselves have generalized after observing several numerical cases with supportive instruction), the transition to the use of literal symbols to represent these ideas is not particularly difficult. Carpenter et al. found that first and second graders, for example, were able to generate representations such as $m + 0 = m$ after they had explored numerical cases illustrating the additive identity and had generated a conjecture stating that "zero plus a number equals that number." Likewise, third-grade students in Blanton, Stephens, et al.'s (2015) study who experienced early algebra instruction involving in part generalized arithmetic were better able to identify a symbolic representation of a fundamental property (in this case, $a - a = 0$) instantiated in numerical examples (in this case, $8 - 8 =$ ___ and $12 - 12 =$ ___) than were students who did not receive the instruction. In fact, research suggests that once students become comfortable with variable notation they may find that some relationships are more easily represented through such notation (Blanton, Stephens, et al., 2015; Brizuela, Blanton, Sawrey, Newman-Owens, & Gardiner, 2015; Carpenter et al., 2003).

*Justifying Arithmetic Relationships*

Proof and justification in school mathematics has received increased attention over the past 15 years, with researchers and policy makers arguing that it must be a central part of the education of all students at all grade levels (Ball, Hoyles, Jahnke, & Movshovitz-Hadar, 2002; Knuth, 2002; NCTM, 2000; NGA & CCSSO, 2010; RAND Mathematics Study Panel, 2003). The abundance of research showing that students in secondary grades and beyond struggle with proof (Stylianou, Blanton, & Rotou, 2015; Usiskin, 1987; K. Weber, 2001) has led to arguments that students' experiences with proof should not be treated as an extraneous topic but rather as an explanatory tool that students begin developing through the construction of informal arguments in the elementary grades (Stylianou, Blanton, & Knuth, 2009).

Although justifying is a practice with which many students still struggle (e.g., Carpenter et al., 2003; Knuth, Choppin, & Bieda, 2009; Knuth, Choppin, Slaughter, & Sutherland, 2002; G. J. Stylianides, Stylianides, & Weber, 2016, this volume), there is growing evidence that students in kindergarten through grade 8 are capable of developing informal mathematical arguments to justify the generalizations they notice. Knuth et al. (2009) identified four levels of thinking students exhibit when developing arguments (summarized in Table 15.3). Level 1 (Knuth et al., 2009) reasoning is extremely common in elementary grades (e.g., Carpenter et al., 2003; Isler, Stephens, Gardiner, Knuth, & Blanton, 2013) and middle grades (e.g., Knuth et al., 2009; Knuth et al., 2002). For

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**TABLE 15.3.** Proof Production Framework

| Proof production level | Characteristics of level |
|---|---|
| 0 | Students are unaware of the need to provide a mathematical justification to demonstrate the truth of a proposition or statement. |
| 1 | Students are aware of the need to provide a mathematical justification, but their justifications are not general; in the majority of cases, they are empirically based. |
| 2 | Students are aware of the need for a general argument but are unable to produce one. They often make an attempt, but the arguments they produce fall short of being acceptable proofs. |
| 3 | Students are aware of the need for a general argument and are able to successfully produce one that demonstrates a proposition is true in all cases. |

*Note.* Based on "Middle School Students' Production of Mathematical Justifications," by E. J. Knuth, J. M. Choppin, and K. N. Bieda, in D. A. Stylianou, M. Blanton, and E. J. Knuth (Eds.), *Teaching and Learning Proof Across the Grades: A K–16 Perspective* (pp. 153–170), 2009, New York, NY: Routledge/Taylor & Francis Group.

example, Isler et al. (2013) found that when third-grade students were posed the task, "Brian knows that anytime you add three odd numbers, you will always get an odd number. Explain why this is always true" (p. 141) on a written assessment, the most common response—both before and after an instructional intervention—involved providing an empirical argument, such as "3 + 5 + 7 = 15, that's an odd number" (p. 142). A. J. Stylianides (2007; also described in Ball & Bass, 2003) observed Level 1 thinking in a third-grade class when students were asked to consider the Two-Coin problem: "I have pennies, nickels, and dimes in my pocket. Suppose I pull out two coins. How much money might I have?" (p. 302). Students generated all the combinations they could find by conducting the experiment with actual coins and concluded they had found them all.

Knuth et al. (2009) found that middle school students responded similarly to tasks concerning algebraic "number tricks" such as the following:

> Mei discovered a number trick. She takes a number and multiplies it by 5, and then adds 12. She then subtracts the starting number and divides the result by 4. She notices the answer she gets is always 3 more than the number she started with. Malaika doesn't think this will happen again, so she tried the trick with another number. Mei and Malaika decide that they will always get a result that is 3 more than the starting number. Do you think they are right? How would you convince a classmate that you would always get a result that is three more than the starting number? (p. 157)

Knuth et al. (2009) found that 78% of sixth-grade students, 79% of seventh-grade students, and 81% of eighth-grade students gave example-based justifications, such as "I would try the strategy a few times to prove it works" (p. 157).

Level 2 reasoning (Knuth et al., 2009) is characterized by an awareness of the need for a general argument but an inability to produce one. Students at this level often acknowledge that the empirical arguments they have produced do not "prove" the claim in question but are unsure how to proceed. In this regard, research suggests that students can sometimes use their strategies in Level 1 thinking to scaffold their movement toward higher levels of thinking. That is, although empirical arguments do not qualify as general arguments, empirical explorations are often useful in helping students identify patterns and generate conjectures (A. J. Stylianides, 2007). Carpenter et al. (2003) described how one second-grade student used special types of numbers ("really large numbers" and "really low numbers") to think about whether the equation $a + b - b = a$ was true or false. As she explored specific cases, she recognized fundamental properties of the operation of addition at play (e.g., additive inverse and additive identity), and in a justification reflecting Level 3 thinking, invoked these properties to build a more general argument. In the case of the Two-Coin problem, the fact that students generated empirical examples offered a starting point from which they later developed valid modes of argumentation by systematically finding all solutions and engaging in "proof by exhaustion" (i.e., engaging in Level 3 reasoning; Knuth et al., 2009), something they did successfully on several follow-up tasks. Ellis, Lockwood, Williams, Dogan, and Knuth's (2012) interviews with middle school students likewise showed that many students who explored conjectures with multiple examples were then able to provide deductive arguments, valid counterexamples, or general arguments to justify or refute the conjectures.

Further evidence that elementary students are capable of moving beyond examples-based reasoning in the domain of generalized arithmetic comes from studies that employed classroom observations or written assessments (e.g., Bastable & Schifter, 2008; Carpenter et al., 2003; Isler et al., 2013; Russell et al., 2011a; Schifter, 2009; Stephens, Blanton, Knuth, Isler, & Gardiner, 2015) to explore students' justifications of properties of operations, generalizations about operations on even numbers

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and odd numbers, and other computational regularities. This work shows that students can learn to develop "representation-based" arguments (e.g., Schifter, 2009) that rely on the use of diagrams, manipulatives, or story contexts as the basis of justifications about the truth of generalizations. The arguments in such cases consist of the representations themselves along with actions students perform with the representations and the explanations they offer to support the truth of the given claims. Importantly, such arguments do not rely on testing numerical cases.

Isler et al. (2013; also described in Stephens et al., 2015), for example, found that third graders who received early algebra instruction that focused in part on conjecturing and justifying in the context of generalized arithmetic were able to produce representation-based arguments about the sums of even numbers and odd numbers. One student, for example, used Unifix cubes to illustrate why the sum of two odd numbers is an even number:

> I did it with blocks. So, I took 9 blocks, and I added it to 11. If you look at the blocks alone, 9 and 11, they each have a leftover, but when you put them together, their leftovers get paired up, so you have an even number. (See Figure 15.1; Stephens et al., 2015, p. 98)

This student's justification is a representation-based argument because, although 9 blocks and 11 blocks were used, the student's explanation and Unifix model suggest that he understood that the model represented any odd number of cubes. Furthermore, the student did not calculate in the process of justifying.

Schifter (2009), likewise, found that first- and second-grade students were able to show why the commutative property of addition holds for all numbers by using cubes to represent addition as the joining of two sets and treating the representation in such a way that—even though a specific number of cubes is necessarily involved—any two whole numbers could be accommodated. The students' actions with the representation then demonstrated that switching the placement of the two quantities did not change the total. As discussed earlier, it is important to interpret such a finding with caution and not attribute thinking to students that may not be present. For example, it is unclear if these students' actions demonstrate an understanding of the commutative property of addition or if they demonstrate an understanding that the total number of cubes does not change no matter how they are organized (i.e., conservation of quantity).

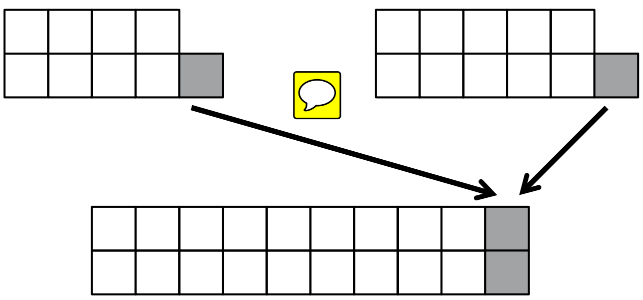

**FIGURE 15.1.** *A student's use of representation-based reasoning to justify why the sum of two odd numbers is an even number. From "Just Say Yes to Early Algebra!," by A. C. Stephens, M. Blanton, E. J. Knuth, I. Isler, and A. M. Gardiner, 2015,* Teaching Children Mathematics, *22(2), p. 98.*

In more formal mathematics, properties of operations, such as the commutative property of addition, are axioms that are assumed without proof. In elementary grades, however, it is important for students to build arguments that explain how these operations work for all numbers. Such activity provides students an opportunity both to deepen their understanding of arithmetic and to engage in the algebraic thinking practice of justifying relationships. Though some students in Schifter's (2009) study may have held deeper understandings than others, they were engaged in important practices appropriate for their grade level.

We conclude this section on generalized arithmetic with two observations. First, we note that the vast majority of studies cited here (with the exception of Empson et al.'s, 2011, work) involved students generalizing, representing, justifying, and reasoning with relationships involving *whole numbers.* Although whole numbers provide a natural context in which to engage students in these algebraic thinking practices in the elementary grades, we are left with the question of what students can do within other numerical domains. Empson et al. (2011) argued for the existence of "conceptual continuities" (p. 410) between whole-number arithmetic and fractions. Their work suggests that if provided with supportive instruction, students can employ their implicit understandings of properties of operations and equality to solve fraction word problems.

Given the difficulties older students experience with both fractions and algebra, more work is needed to explore the opportunities offered by the understandings students bring from a deep understanding of whole number as well as the possible limits of those understandings. Is understanding the commutative property of addition, for example, much more complex when students are working with fractions? Do students understand that a variable that stands for *any* number could stand for a decimal or fraction, or do they have a natural number bias (as Christou and Vosniadou, 2012, found with high school students)? When students make a representation-based argument with whole numbers to justify a claim, do they really understand that the claim is also true in the case of fractions? Finally, are more complex number domains part of the story when we think about the difficulties students in the secondary grades experience with algebra, or can the vast majority of students' difficulties be attributed to deficiencies in their learning of whole-number arithmetic?

Second, we note that in contrast to recent research in the elementary grades, there appear to be fewer classroom "success stories" about the engagement of students with mathematical justification in the middle grades. Why is this the case? It should be emphasized that the research illustrating elementary grades students' productive engagement with generalizing, representing, justifying, and reasoning with arithmetic relationships was in the majority of cases conducted in the context of very supportive classroom environments with knowledgeable teachers or teacher-researchers who valued the development of students' abilities to engage in these practices. Mathematical argumentation is a complex activity that requires nuance as teachers and students navigate the meaning of "proof" in their classroom contexts. It is also an activity that cannot be divorced from the process of conjecturing (Garuti, Boero, & Lemut, 1998) or generalizing (Ellis, 2007a). The literature reviewed here suggests a tight link in elementary grades interventions between generalizing and justifying, with students first exploring mathematical statements with numerical cases and making conjectures before being asked to justify a generalization (e.g., Bastable & Schifter, 2008; Blanton, Stephens, et al., 2015; Carpenter et al., 2003; Russell et al., 2011a, 2011b; Schifter, 2009).

There is much less research at the middle school level addressing students' abilities to justify arithmetic relationships. The studies that do exist document students' thinking in the context of their everyday instruction (Knuth et al., 2002, 2009) rather than in relation to a supportive instructional intervention. Bieda (2010) found that middle school teachers often do not take full advantage of opportunities to engage their students in justification. An analysis of a popular middle school mathematics curriculum furthermore suggests that, although students are often asked to offer rationales for their solutions, there is little support offered to teachers in implementing the opportunities for justification present in the instructional materials (G. J. Stylianides, 2009; see also G. J. Stylianides et al., 2016, this volume). Thus, more research is needed concerning what middle grades students can do when provided rich opportunities to explore arithmetic relationships and to build conjectures before engaging with the activity of justifying (see G. J. Stylianides et al., 2016, this volume, for a similar call for intervention-oriented studies in the area of proof).

## Functional Thinking

### *Functions as a Context for Algebraic Thinking*

Although formal algebra courses have traditionally used a transformational approach emphasizing literal symbols, expressions, and equations (Kieran, 2007), research documenting misconceptions and lack of structural understanding of these mathematical objects among adolescent-and-older students (e.g., Herscovics & Linchevski, 1994; Huntley, Marcus, Kahan, & Miller, 2007; Stacey & MacGregor, 1999) has led some to suggest that functions might serve as a better organizing concept for teaching and learning algebra (e.g., Dubinsky & Harel, 1992; Schwartz & Yerushalmy, 1992). Arguments for this approach include the idea that functions can unite a wide range of otherwise isolated topics, such as operations on numbers, fractions, ratio and proportion, and formulas relating quantities (Carraher, Schliemann, & Schwartz, 2008); the observation that functions can serve as a connection between students' day-to-day experiences and mathematics (Chazan, 2000); and the finding that such an approach naturally encourages student inquiry (Yerushalmy, 2000). Perhaps more important for the purposes of this chapter, functional thinking provides a rich context for developing the algebraic thinking practices of generalizing, representing, justifying, and reasoning with relationships between quantities (Blanton et al., 2011; Carraher & Schliemann, 2007).

Recent studies in secondary grades documenting students' difficulties with a more formal study of functions (e.g., Bush & Karp, 2013; Huntley et al., 2007; Knuth, 2000; Lobato et al., 2003) raise the question, however, of whether the study of functions in kindergarten through grade 8 would be a suitable context for developing students' algebraic thinking. Nevertheless, the growing body of evidence that K–grade 8 students *can* successfully reason algebraically about functional relationships opens the possibility that the difficulties exhibited by older students might stem from a lack of experiences with functional thinking in the elementary grades. Carraher, Schliemann, and Schwartz (2008) argued more generally about transitions into the formal study of algebra:

> The fact that most students throughout the United States do not make this transition easily, nor early, may well say more about our failure to offer suitable conditions for them to learn algebra as an integral part of *elementary* [emphasis added] mathematics than it does about the limitations of their mental structures. (pp. 268–269)

Two perspectives are generally taken in a functions approach to school algebra—a correspondence perspective and a coordination/covariation perspective. Both are concerned with the nature of relationships between quantities. Different researchers emphasize aspects of quantity in different ways. In some traditions of research, *quantity* refers inclusively to a "property of a phenomenon, body, or substance, where the property has a magnitude that can be expressed as a number and a reference" (VIM3: International Vocabulary of Metrology as cited in Carraher & Schliemann, 2015, p. 193). In the context of functional relationships discussed in this section, mathematical relationships are between values that are associated with quantities, where such quantities might be number of items, length, time, or even derived quantities such as speed (Carraher & Schliemann, 2015). Other researchers, whose work is discussed in depth in the Quantitative Reasoning section, emphasize quantities as mental constructions composed of one's conception of an object, a quality of the object, an appropriate unit or dimension, and a process for assigning a numerical value to the quality (e.g., Castillo-Garsow, Johnson, & Moore, 2013; Ellis, 2007b; J. Smith & Thompson, 2008; Thompson, 1994; Thompson & Thompson, 1992). Thompson (2011) highlighted quantification as a process of "conceptualizing an object and an attribute of it so that the attribute has a unit of measure, and the attribute's measure entails a proportional relationship (linear, bi-linear, or multi-linear) with its unit" (p. 37). Quantitative operations are therefore *conceptual operations* by which one conceives of quantities in relation to one or more already conceived quantities.

Both the correspondence and the coordination/covariation perspectives have affordances in developing students' algebraic thinking—affordances that might shift in their significance at different grade bands. Our goal here is to highlight research around both of these perspectives rather than advocate a particular approach. A correspondence perspective, traditionally a more common way to start algebra (Yerushalmy, 2000), involves developing a closed-form rule to describe a relationship between quantities (Confrey & Smith, 1994) that can be used to analyze and predict function behavior. Such rules are also useful because they allow one to determine information about a specific function value without knowing other function values.

On the other hand, Confrey and Smith (1994) described what they called a covariation perspective (also described as the coordination perspective; e.g., Thompson and Carlson, 2016, this volume), which involves examining functions in terms of coordinated changes of $x$- and $y$-values. Confrey and Smith described the covariation approach as including the ability to "move operationally from $y_m$ to $y_{m+1}$ coordinating with movement from $x_m$ to $x_{m+1}$. For tables, it involves the coordination of the variation in two or more columns as one moves down (or up) the table" (p. 137). In middle grades, this type of covariational thinking can build organically from ways in which students notice relationships, especially when the independent variable is an indexing variable such as time. Covariational thinking's focus on coordinating change in $y$ with change in $x$—a concept that has growing significance as students advance into more formal mathematics in middle grades—can help students develop an understanding of classes of functions in terms of their characteristic action (e.g., linear functions involve constant rates of change, quadratic functions involve rates of change that are constantly changing; Confrey & Smith, 1994). Although we address research on both the correspondence and covariation perspectives here, we revisit the covariation perspective in the next section (see the Quantitative Reasoning section in this chapter).

### *Generalizing and Reasoning With Functional Relationships*

Functions studied in K–grade 8 mathematics are often well behaved (e.g., are perfectly linear, even if this makes the situation somewhat less realistic) and have an underlying closed-form mathematical rule. Although "real-world" data have powerful affordances in mathematical modeling, particularly in later grades, "clean data" is often used in the elementary grades and may be more productive in the middle grades because this type of data allows students to focus on the practices of generalizing and justifying relationships (Ellis, 2007a).

Although the bulk of research on students' functional thinking in kindergarten through grade 8 has involved linear relationships, some studies indicate that students can successfully generalize nonlinear relationships as well. In what follows, we examine research on both function types.

**Linear relationships.** By the mid-2000s (e.g., Lester, 2007), there was mounting evidence that students in the elementary and middle grades are capable of generalizing linear relationships. Recent research (e.g., Blanton, Brizuela, Gardiner, Sawrey, & Newman-Owens, 2015; Blanton, Stephens, et al., 2015; Carraher, Martinez, & Schliemann, 2008; Cooper & Warren, 2011; Ellis, 2007a; Lannin, Barker, & Townsend, 2006; Martinez & Brizuela, 2006) continues to demonstrate that elementary and middle school students can attend to two quantities that vary together, describe one quantity in relation to the other, understand input-output rules, and identify correspondence relationships.

A focus of some of this recent research was the distinction between recursive and functional thinking and how to move students from the former to the latter. For example, Carraher, Martinez, and Schliemann (2008) asked third-grade students to consider the case of a series of square dinner tables that can each seat four people. Students were asked to complete a function table showing the relationship between the number of dinner tables and the total number of people that could be seated. They were then asked to consider a scenario in which the tables are pushed together in a row, thus allowing just two people to sit at each table plus one person at each end of the row. Reasoning with function tables constructed to represent problem data, many students noticed the constant increase in the dependent variable column but struggled to construct a correspondence rule relating the two quantities. As Martinez and Brizuela (2006) described, some students adopted a "hybrid approach" that included simultaneous attention to both recursive and functional features of the relationship but still did not result in a correspondence rule. Carraher, Martinez, and Schliemann (2008) noted that function tables in which the independent values increase by one might focus students' attention on the recursive pattern and, thus, explain why students had difficulties with producing a correspondence rule to describe the functional relationship. They argued, however, that such tables are valuable in that they "permit a visual scanning of results that can be helpful for students to grasp how the function 'works'" (p. 18) before they focus on a relationship between the two quantities. Some have suggested that using function tables that are not well-ordered or asking students to state the dependent variable for a large or unknown value of the independent variable (e.g., How many people can be seated at *t* dinner tables?) may encourage students to look "across" the rows in a function table rather than "down" the columns in the table to generalize the relationship between the quantities (Schliemann et al., 2007; Warren, Cooper, & Lamb, 2006).

Although much of this research has occurred in upper elementary grades (grades 3–5), recent research provides evidence that students as early as first grade (age 6) are able to generalize and reason with functional relationships in numerical contexts. Blanton, Brizuela, et al. (2015) conducted classroom teaching experiments that focused on generating function data from problem situations involving linear relationships of the form $y = mx$, $m \in \mathbb{N}$, and $y = x + b$, $b \in \mathbb{N}$, organizing the data in function tables, exploring relationships in the data, and using the tables to both predict near and far function values and generalize relationships using both words and variable notation. Their key result—a trajectory describing the levels of sophistication with which first-grade students generalized functional relationships (see Table 15.4)—suggests that young children can reason about functional relationships in surprisingly sophisticated ways.

Another interesting finding from this work is that first-grade students exhibited fewer challenges with issues prevalent in the literature on the thinking of older students, including the above-mentioned difficulty in shifting from recursive to functional thinking. Although Carraher, Martinez, and Schliemann (2008) stated about their third-grade study, "It may be too much to hope that students will learn to express formula-based linear functions straightaway through closed expressions" (p. 18), Blanton, Brizuela, et al. (2015) found that first-grade students were able to construct such representations and did not necessarily need to engage with recursive thinking before developing closed-form rules. Although this seems to contrast with what they and others have generally found in upper elementary grades (e.g., Blanton, Stephens, et al., 2015; Carraher, Martinez, & Schliemann, 2008) and with the argument that recursive thinking might be an important precursor to thinking about relationships between quantities (Carraher, Martinez, & Schliemann, 2008; Rivera & Becker, 2011), Blanton, Brizuela, et al. (2015) suggested that children at the start of formal schooling have not yet spent an extensive amount of time studying recursive patterns and, as a result, may have more flexibility in thinking about correspondence relationships between quantities. They also maintained that systematically representing a relationship between two *specific* related values with an

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**TABLE 15.4.** Levels in 6-Year-Olds' Understandings of Functional Relationships

| Level | Characteristics |
|---|---|
| Pre-structural | Does not describe or use (even implicitly) any mathematical relationship in talking about problem data. May notice a non-mathematical regularity in the inscriptions. |
| Recursive-particular | Conceptualizes a recursive pattern as a sequence of particular instances. |
| Recursive-general | Conceptualizes a recursive pattern as a generalized rule between arbitrary successive values. |
| Functional-particular | Conceptualizes a functional relationship as a sequence of particular relationships between specific corresponding values. |
| Primitive functional-general | Conceptualizes a functional relationship as a general relationship between two quantities, but cannot describe a mathematical transformation on two arbitrary quantities. |
| Emergent functional-general | Conceptualization of functional relationship reflects emergence of key attributes (e.g., characterizing the generalized quantities or mathematical transformation). |
| Condensed functional-general | Conceptualizes a functional relationship as a generalized relationship between two arbitrary and explicitly noted quantities. |
| Function as object | Perceives boundaries in the generality of the functional relationship; conceptualizes functional relationship as object on which operations could be performed. |

*Note.* Based on "A Learning Trajectory in 6-Year-Olds' Thinking About Generalizing Functional Relationships," by M. Blanton, B. Brizuela, A. Gardiner, K. Sawrey, and A. Newman-Owens, 2015, *Journal for Research in Mathematics Education* 46(5), 511–558.

equation seemed to scaffold students' ability to generalize a relationship between variable quantities.

Research that examines students' functional thinking typically begins by engaging students in some type of physical or conceptual activity (i.e., a "problematic"; E. Smith, 2008) in which two or more quantities are related. One such activity through which functional thinking can develop is the examination of visual patterns. Rivera (2010a) found that, prior to formal pattern instruction, second-grade students were able to attend to the linear relationships between two quantities as they examined a figural pattern in which the relationship was visually embedded. Rivera (2010b) distinguished a *figural pattern* from a geometric sequence, where the former "consists of stages whose parts could be interpreted as being configured in a certain way" (pp. 297–298). Even though participants were not asked to identify function rules (in either words or variable notation), they implicitly attended to how two quantities were related as they determined the values of dependent variables given particular values of independent variables through their analysis of figural patterns. Students were not able to do so with a nonfigural function task, leading Rivera (2010a) to join several other researchers (e.g., Lannin, 2005; Moss & McNab, 2011) in proposing that tasks involving visual growing patterns support students' early attempts to generalize relationships between two quantities.

In a similar line of inquiry, Warren and Cooper (2008) found that after two lessons, third-grade students could begin to develop language linking patterns to position numbers of figures and generalize a relationship between the position number and number of objects in the related figure (e.g., "it's double the number of the step"). Students were supported by the use of concrete materials, such as tiles to create geometric patterns (or what Rivera refers to as figural patterns) and position cards to draw attention to the independent variable, as well as color to represent different components of a pattern. Moss and McNab (2011) likewise found that second-grade students were successful in generalizing visual patterns by focusing on position numbers as independent variables as opposed to a recursive pattern (i.e., variation in a single sequence of values; Blanton et al., 2011) in the dependent variable. They found that students' work with visual patterns led to spatially inspired terms such as "bump" for the *y*-intercept that helped them successfully generalize purely numeric patterns.

Middle school students have also demonstrated some success in generalizing relationships conveyed through visual patterns. Lannin (2005) found that function contexts that involve figural patterns allowed sixth-grade students to visualize changes in a relationship and connect their generalizations to a visual representation, a process that may have led to increased success for some students (see also Rivera & Becker, 2005). In a 3-year longitudinal study on the development of middle school students' generalization of functional relationships in the context of visual growing patterns, Rivera and Becker (2011) found that students initially tended toward visual strategies, later moved toward numerical strategies (e.g., looking for relationships in a function table with-

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out considering the visual pattern it represents), then returned to visual strategies as the teaching experiment emphasized the development of multiplicative thinking. Rivera and Becker found that, although students' use of numerical strategies sometimes simplified the process of constructing direct formulas, these formulas were at times problematic and difficult to justify. Visual strategies were more powerful than numerical strategies alone for identifying functional relationships, especially when students' understanding of multiplication as "groups of" was strengthened (Rivera & Becker, 2011).

Despite this growing collection of research illustrating that even young children can engage with important mathematical ideas regarding linear relationships, many do not have the opportunity to do so. Although the Common Core State Standards for Mathematics (NGA & CCSSO, 2010), for example, do include important early algebraic ideas (e.g., generalized arithmetic) in the elementary grades, the lack of attention these standards—as well as curricular materials—give to functional thinking prior to formal algebra in the later grades seems to be an opportunity missed. As Carraher, Schliemann, and Schwartz (2008) asserted, "It is nothing short of remarkable that the topic of functions is absent from early mathematics curricula" (p. 265).

**Nonlinear relationships.** There is evidence that middle grades students can also generalize quadratic and exponential relationships (Ellis, 2011a, 2011b; Francisco & Hähkiöniemi, 2012; Lobato, Hohensee, Rhodehamel, & Diamond, 2012). Ellis (2011b), for example, had eighth-grade students explore relationships among rectangle lengths, widths, and areas using dynamic geometry software that led to collective generalizing about the patterns they saw in the data concerning changing rates of growth and constant second differences. Students initially took what Ellis called a covariation perspective, in which they coordinated the growth of the height, length, and area of a rectangle, and then extended this reasoning to develop a correspondence rule. Ellis (2011a) argued that this shift from a covariation to a correspondence view was aided by students' abilities to attend to the relevant quantities of height, width, and area. Students in these later grades did not have difficulty working with function tables that were not well ordered, for example, because their thinking was grounded in the imagery of the rectangle's dimensions. Ellis's work further illustrates that the act of generalizing can be framed as a situated act influenced by social interactions, tools, tasks, and classroom norms and promoted by encouraging justification or clarification and focusing attention on mathematical relationships (Ellis, 2011b).

Elsewhere, Francisco and Hähkiöniemi (2012) investigated seventh-grade students' thinking across several mathematical content strands in a 2-year, after-school, classroom-based research project. Six sessions were devoted to using "Guess-My-Rule" games with quadratic functions. These games, in which students were given a function table of data points, encouraged students to develop different types of function rules to describe the data. They found that many students began their investigation of the values in the function table by focusing on recursive relationships. Ian, for example, noticed the symmetry of the function $y = (x - 1)^2$ as shown in a function table and began computing differences between successive $y$ values shown in the table. He noticed that, starting at zero, $y$ values increased by 1, 3, 5, and so on. When asked for an explicit rule, he said:

> I got it but. I just got it. Look, 4 times 4 minus 7 equals 9 [writes $4 \times 4 - 7 = 9$]. Look. Then, if you do the next 3 times 3 minus 5 [writes $3 \times 3 - 5 = 4$]. . . . Look. I got it. It's right there! [Adds $2 \times 2 - 3 = 1$]. I just don't know what the rule is. It's $x$ times $x$ minus an odd number. (p. 1012)

Ian's observation that he could obtain $y$ values from $x$ values by multiplying $x$ by itself and then subtracting an odd number that decreased by two every time $x$ decreased by one indicated he was able to generalize the relationship and use variables to represent indeterminate quantities. His explicit rule, however, still included a recursive sub rule, illustrating again the links students sometimes make between these different ways of interpreting function data through a "hybrid" approach reminiscent of that found by Martinez and Brizuela (2006) in elementary grades. As already mentioned, however, Blanton, Brizuela, et al. (2015) found that first-grade students exhibited a fluency with noticing and representing functional (linear) relationships that was not hindered by a dependence on recursive thinking, lending further support to the argument that introducing children to functional relationships at the start of formal schooling might offset an overreliance on recursive thinking that can impede their understanding of covarying relationships in later grades.

### *Representing Functional Relationships*

E. Smith (2008) notes that, after students identify relationships in a functional situation, they are often asked to create a representation of the relationship. Such representations can take many forms, including function tables, coordinate graphs, pictures, and algebraic

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equations. Representations do much more than register what students understand—they can also help students structure and extend their thinking, with different representations highlighting different aspects of information while hiding others (Blanton & Kaput, 2011; Brizuela & Earnest, 2008; Caddle & Brizuela, 2011). For example, function tables provide discrete points of information that are more "hidden" in a graph or equation; however, coordinate graphs provide a more holistic perspective on a function, can visually highlight concepts such as slope and points of intersection, and can allow students to deal with functions as objects (Schwartz & Yerushalmy, 1992). Teaching functions in an integrated fashion, where multiple representations are used consistently, encourages the use of a variety of methods to solve problems involving functional thinking (Bush & Karp, 2013; Duval, 2006).

As we did in our discussion of representing arithmetic relationships, we focus again on research concerning students' use of variable notation as a way to symbolize or represent functions. Students might first experience variable in the role of varying quantity when they are asked to work with variable quantities. Carraher, Schliemann, and Schwartz (2008) found that grades 2–4 students could successfully use variable notation to represent the amount of candy two children had, where each child had a box with an identical number of candies and one child had an additional three candies on top of her box. Although students initially assigned particular values to the number of candies the two children had, through classroom discussion they became comfortable with the use of literal symbols to represent varying, unknown quantities.

Blanton, Stephens, et al. (2015) posed a similar task—the Piggy Bank problem (see Figure 15.2)—to third-grade students on an assessment given before and after a 1-year early algebra intervention. They found that many students were unable to respond to these questions at pretest and that those who did overwhelmingly chose specific numerical values for the number of pennies Tim and Angela each had. No students represented these indeterminate quantities with variable notation. By posttest, however, three-quarters of the students used variable expressions to represent the number of pennies Tim and Angela each had, and over half did so to represent the total number of pennies. Furthermore, the majority of these students generated representations conveying that Tim and Angela had the same number of pennies in their bank—if $b$ represented the number of pennies in Tim's bank, then Angela's number of pennies could be represented by $b + 8$—and their combined number of pennies could be represented by $b + b + 8$.

**The Piggy Bank Problem**

Tim and Angela each have a piggy bank. They know that their piggy banks each contain the same number of pennies, but they don't know how many. Angela also has 8 pennies in her hand.

(a) How would you describe the number of pennies Tim has?
(b) How would you describe the total number of pennies Angela has?
(c) Angela and Tim combine all of their pennies to buy some candy. How would you describe the total number of pennies they have?

**FIGURE 15.2.** *Piggy Bank problem. From "The Development of Children's Algebraic Thinking: The Impact of a Comprehensive Early Algebra Intervention in Third Grade," by M. Blanton, A. Stephens, E. Knuth, A. Gardiner, I. Isler, and J. Kim, 2015,* Journal for Research in Mathematics Education, 46*(1), p. 59.*

As noted earlier, although researchers have traditionally assumed that students' experiences with variable notation should be reserved until they have had extensive experience expressing generalizations with words, some early algebra researchers now question withholding this notation (Blanton, Brizuela, et al., 2015; Brizuela & Earnest, 2008; Carpenter et al., 2003; Russell et al., 2011a) and argue instead that students should be given the opportunity to use variable notation from the start of formal schooling (Brizuela et al., 2015). Recent research even suggests that students might choose variable notation over the use of words as a more succinct way to express functional relationships and, given a working knowledge of variable notation, that they can spontaneously produce such representations. In a quasi-experimental study, Blanton, Stephens, et al. (2015; see also Isler et al., 2014) found that third-grade students participating in a 1-year early algebra intervention were more successful representing a linear relationship of the form $y = mx$, $m \in \mathbb{N}$, with variable notation than with words.

Elsewhere, Blanton, Brizuela, et al. (2015) found that first-grade students could represent functional relation-

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ships using variable notation and develop an emergent understanding of variable as a varying quantity that could be treated as an object in reasoning with generalized forms. In a related study of four first-grade students' understandings of variable, Brizuela et al. (2015) found that the students demonstrated various misconceptions about variable also exhibited by older students—including that variable notation is an object or label, that the value of the variable is related in some way to the literal symbol's position in the alphabet, and that literal symbols and numbers should not be combined in a single equation. However, one finding that held across all four students is the idea that variable notation represents indeterminate quantities. Brizuela et al. (2015) explained that although students typically searched for specific values that could instantiate the quantities in the situations they were discussing, when variable notation was used students had no need to refer to specific values. This suggests that children's understanding of variable notation provided them a mechanism for representing general quantities and their relationships. Brizuela et al. (2015) argued that variable notation acted as a mediating tool (Kaput et al., 2008) to facilitate children's reflections about indeterminate quantities and that conceptual understandings do not necessarily need to precede the introduction of symbols such as variable notation, but rather that meanings and symbols can co-emerge.

Middle grades students have demonstrated success representing functional relationships with variable notation as well, including linear relationships (e.g., Rivera & Becker, 2011), quadratic relationships (e.g., Ellis, 2011b; Francisco & Hähkiöniemi, 2012), and exponential relationships (e.g., Ellis et al., 2013). In the study led by Francisco and Hähkiöniemi (2012), one pair of students, Jerel and Chris, was tasked with investigating a function table representing the function $y = (x + 1)^2$ to find a rule. Jerel explained his rule: "It is $x$ plus one, and then you multiply. I mean, then, you time the sum. I mean then you get the sum, and then you time the sum . . . by the sum" (p. 1014). They struggled, however, with how to write the rule using literal symbols. Chris called the expression $(x + 1)$ "the sum," while Jerel called it "the new $x$." Chris wrote his rule as $x + 1 \times (\text{sum}) = y$. The ensuing conversation illustrated that they were thinking of their rule as a composite one, where $x + 1 = z$ and $z \times z = y$. With some assistance from the researcher, they rewrote their rule as $(x + 1) \cdot (x + 1) = y$. Although this illustrates that middle school students can work with multiple variables and view the generalized relationships—functions—they represent as objects of reasoning, it does raise the question of whether these students would have had more ease with this process if their experiences in elementary grades had included a more sustained study of functions and their representations.

### *Justifying Functional Relationships*

Researchers have argued that justifying—characterized by E. Smith (2008) as seeking mathematical certainty—is tightly linked to the activity of generalizing (Ellis, 2007a; Lannin, 2005; Rivera, 2010b) and interacts with and drives the symbolization process (Kaput et al., 2008). As alluded to earlier, the literature is replete with evidence that students tend to justify generalizations by providing empirical examples that "fit" the generalization (e.g., Lannin, 2005; Lannin et al., 2006). In the context of functional thinking, this might entail pointing to an ordered pair of values in a function table that fits a hypothesized function rule as "proof" that the rule describes the relationship. But students can also justify functional relationships by interpreting the symbolizations they produce in terms of the quantities being compared.

For example, 6-year-olds in Blanton, Brizuela, et al.'s (2015) study were able to justify the functional relationships they found by reasoning with the given problem situation. In one task, students were asked to consider the relationship between the number of train stops and the number of cars on the train if a train stops at every station along its route and picks up two train cars at each station (assuming the engine was not counted and the train began its route with only the engine). One student explained her representation, $R + R = V$, where $R$ represented the "number of stops" and $V$ represented the "number of cars" as "Whatever number, how many stops it made, if you doubled it, that's how many cars it would have" (p. 536). Although this student used relationships she had observed in the function table she generated to generalize the relationship, she was also able to connect the quantities—number of stops and number of train cars—present in the problem context to justify the relationship she noticed.

Working with linear data, middle school students in van Reeuwijk and Wijers' (1997) study were able to provide justifications explaining the origins of their generalizations in terms of the related quantities. For instance, one group of students generalized that to build a hall of length $L$, they would need to multiply the length

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by three and then add on one less than the length to that product to determine the number of beams needed. They formalized their generalization by writing $L \times 3 + L - 1$. The students were able to explain this formula in terms of the quantities as follows: "For length $L$, we have $L$ triangles, which have $L \times 3$ rods, and then we need another $(L - 1)$ rods for the top of the beam" (p. 232).

A common theme in these studies is that the function tasks are situated in problem scenarios that seemed to productively support students' justifications through visualization. This finding relates back to Rivera and Becker's (2011) research regarding middle grades students' work with visual patterns. Recall that students were more successful generalizing linear visual patterns when their strategies were tilted toward figural approaches. Rivera (2007, 2010b) likewise found that students were more successful justifying their generalizations when a figural approach was used and that students who relied heavily on numerical reasoning alone had difficulty explaining why their formulas described the given scenarios, sometimes confusing construction with justification (Rivera & Becker, 2009).

Finally, although students are often asked to generalize and perfect their generalizations before engaging in justification as a final step in the process, Ellis (2007a) found that middle school students benefitted from engaging in an iterative cycle of generalizing *and* justifying. Students' first attempts at generalizations were often limited, or even inaccurate, but through the act of explaining their generalizations and attempting to justify them, students revisited and refined their original generalizations. Ellis (2007a) suggested that teachers should incorporate justification early in the instructional sequence as a way to help students generalize more effectively rather than view justification as an act that follows students' final generalizations.

## Quantitative Reasoning

As discussed in the Functional Thinking section, different traditions of research reference the ideas of quantities and quantitative reasoning in different ways. In this section, we discuss a research tradition that emphasizes quantities as mental constructions. In particular, Thompson and colleagues (Thompson, 1994, 2011; Thompson & Carlson, 2016, this volume; Thompson & Thompson, 1992) described quantities as *conceptual entities*—schemes—composed of a person's conception of an object (such as a piece of rope), a quality of the object (such as its length), an appropriate unit or dimension for measurement (such as inches), and a process of assigning a numerical value to the quality. Length, area, speed, and temperature are examples of attributes that one could measure as quantities. J. Smith and Thompson (2008) emphasized that it is one's capacity to measure, regardless of whether those measurements have been carried out, that make such attributes quantities.

Researchers have argued that algebraic thinking can be developed through students' exploration of quantities and quantitative relationships (Fujii & Stephens, 2008; Olive & Çağlayan, 2008; Steffe & Izsak, 2002). Quantitative reasoning can be characterized as mentally operating on either specified or unknown quantities in a situation to create new quantities and to construct relationships between quantities (Johnson, 2012; Thompson, 1994). In reasoning quantitatively, one can combine and compare quantities either additively (for example, by asking how much taller one person is in relation to another) or multiplicatively (for example, by asking how many times longer one length of rope is than another; Lobato & Siebert, 2002; Thompson, 1988). The associated arithmetic operations could be subtraction and division. Although it draws heavily on everyday experience, quantitative reasoning is eventuated not by reasoning with real-world situations in and of themselves but by the manner in which a student interacts with a given situation. Thus, a student could attend to number patterns extracted from a real-world situation and be engaged in number pattern reasoning alone or could think about the relations between quantities in an imaginary situation not based on a realistic context and be engaged in quantitative reasoning (Ellis, 2007b; Thompson, 1994; Thompson & Thompson, 1992).

### *Generalizing and Reasoning With Quantitative Relationships*

The definition of algebraic reasoning as "quantitative reasoning about constant and varying unknowns" (Steffe & Iszak, 2002, p. 1164) emphasizes the importance of reasoning about quantities that vary. In this section, we explore that form of reasoning more closely in terms of research on K–grade 8 students' generalizing and reasoning with quantities and the relationships between them. Our discussion focuses on the elementary grades using research in which children generalize and reason with nonspecified quantities as a basis for understanding number. We then turn to research in middle grades on students' covariational reasoning about relationships between quantities to examine a dynamic approach to algebraic thinking.

**Generalizing quantities to build number relationships.** One area of quantitative reasoning in the elementary grades is in an early algebra approach developed by Davydov (1991) that suggests children can reason about scalar quantities such as the length, area, volume, and weight of real objects to construct the algebraic structure of the real number system (Schmittau, 2011). This approach enables students to notice, represent, and reason with generalizing structures, including foundational properties of operations involving associativity, commutativity, and inverse. It presumes that the concept of quantity is prior to that of number and, accordingly, involves the comparison of relationships between nonspecified quantities before the introduction of number (Carraher & Schliemann, 2007; Davydov, 1991; Dougherty & Slovin, 2004).

By the time children begin to measure and numerically quantify attributes, they have already established generalized properties for any such quantities. Within Davydov's (1991) approach, algebra is learned not as a generalization of arithmetic but as a generalization of the relationships between quantities. One of algebra's strengths, as found also by Britt and Irwin (2011), is that students' activities of comparing and joining quantities without being distracted by counting items foregrounds the relational aspect of the equals sign over an operational perspective that elicits a "solve this now" approach in students' thinking (Venenciano & Dougherty, 2014).

Many aspects of Davydov's (1991) approach with younger children are consistent with the Dutch realistic mathematics education (RME) movement, as both enable children to build up their reasoning from working with experientially real quantities. Within this movement, "realistic" refers to problems that can be either from the real world or from the imaginary world (Presmeg, 2003). Students are offered problem situations they can imagine and visualize so that the problems are experientially real in the student's mind (van den Heuvel-Panhuizen & Drijvers, 2014). Freudenthal (1977) emphasized the importance of context in teaching and learning mathematics and introduced the notion of progressive formalization, or mathematization. Progressive formalization is a central process within RME and involves children informally exploring mathematical relationships and then gradually progressing to more formal thinking through guided reinvention. Gravemeijer (1999) described three broad levels of progressive formalization: (1) informal, in which children may represent mathematical principles but in ways that lack formal notation or structure; (2) preformal, in which children develop models that are potentially generalizable across many problems; and (3) formal, in which children are able to develop mathematical abstractions, representations, and abbreviations, often far removed from contextual cues.

**Covariational reasoning.** As students progress into upper elementary and middle school, they can leverage their reasoning about relationships between quantities to begin to think covariationally about functional relationships. Thompson and colleagues (Johnson, 2012; Saldanha & Thompson, 1998; Thompson, 1994; Thompson & Carlson, 2016, this volume; Thompson & Thompson, 1992) and E. Smith and Confrey (Confrey & Smith, 1994; E. Smith, 2003) have both discussed covariational reasoning, although in different ways. As discussed in the Functional Thinking section, E. Smith and Confrey described a covariation approach as being able to move operationally from $y_m$ to $y_{m+1}$ while coordinating with movement from $x_m$ to $x_{m+1}$ (Confrey & Smith, 1994; E. Smith 2003). This way of framing covariation as the coordination of two connected sequences describes how students can form and operate with tables of data, understanding that quantities have sequences of values. This conceptualization of covariation can apply to a variety of situations in which students' quantitative images may be static.

Children can also reason about quantities *varying;* in other words, isolating quantities that change together simultaneously and interdependently (Johnson, 2012; Saldanha & Thompson, 1998). This form of reasoning involves mentally coordinating two varying quantities while attending to the ways they change in relation to each other (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002). Castillo-Garsow (2013) characterized this way of reasoning as the imagining of two quantities changing together; students imagine how one variable changes while simultaneously imagining changes in the other. Although this dynamic perspective of covariation can be discrete or continuous, a *continuous* dynamic perspective of covariation involves the coordination of continuous change in one quantity with continuous change in another. For example, imagine a fixed-height rectangle whose length grows continuously, rather than in discrete chunks. As the length grows, the rate of change of the rectangle's area as compared to the growth in length is constant (Johnson, 2012; Saldanha & Thompson, 1998). Johnson (2012) noted that there are three perspectives of covariation—static, discrete dynamic, and continuous dynamic—which emphasize different ways in which students can perceive relationships between covarying quantities.

Research suggests that middle school students' initial ways of reasoning about problems with functional relationships often arise from a coordination or covariation perspective (e.g., Confrey & Smith, 1994). Thus, situating functions as a way to represent the covariation of quantities could be a powerful approach for fostering students' abilities to think about functions in terms of rates of change (Slavit, 1997). As Carlson and Oehrtman (2005) suggested, leveraging situations involving quantities that students are able to manipulate, visualize, and investigate can foster students' abilities to reason flexibly about dynamically changing events. As such, a covariation approach could support a view of mathematics as a way of making sense of the algebraic relations of dependence, causation, interaction, and correlation between quantities (Chazan, 2000).

Recent research suggests that quantitative reasoning in the middle grades can provide a rich context for the core algebraic thinking practices of generalizing and reasoning with functional relationships. Until recently, middle grades students' covariational thinking as a context for algebraic thinking has not been an explicit object of study, but the past few years have seen an emerging body of work investigating situations in which middle grades students explore covarying quantities as they begin to make sense of rates of change and functions. These studies, which all report on findings from small-scale teaching experiments, suggest that allowing students to reason quantitatively about covarying relationships can support students' emerging understanding of linear, quadratic, and exponential functions as students transition into a more formal study of algebra (Ellis, 2007b, 2011a; Ellis & Grinstead, 2008; Ellis, Ozgur, Kulow, Williams, & Amidon, 2012, 2015; Ellis et al., 2013; Johnson, 2011, 2012; Lobato & Siebert, 2002; E. Weber, Ellis, Kulow, & Ozgur, 2014). For instance, a recent teaching experiment about middle school students' understanding of exponential growth explored their thinking as they reasoned with the covarying quantities height and time while investigating the growth of a plant (Ellis et al., 2015). The study's findings suggest that situating an investigation of exponential growth in a model of covarying quantities can support students' understanding of what it means for data to grow exponentially, their ability to express exponential growth relationships algebraically, and their ability to make sense of noninteger exponents.

In studying middle school students' understanding of linear functions, Olive and Çağlayan (2008) explored the unit coordination arising from situations involving systems of linear equations, systems that entail complex quantitative reasoning relating groups of quantitative relationships. Olive and Çağlayan found that identifying and coordinating the units involved in the problem situation were critical aspects of quantitative reasoning. Similarly, Hough and Gough (2007) and van Reeuwijk (2001) reported design experiments relying on alternate methods for working with simultaneous equations that relied on mathematizing quantitative contexts such as items and prices. Van Reeuwijk reported on a unit of instruction called Comparing Quantities, in which students were immersed in realistic situations in which they had to find the values of combinations of items for shopping problems. Van Reeuwijk found that these realistic contexts enabled students to progress to preformal levels, in which students could develop potentially generalizable models and develop concepts related to the algebra of simultaneous equations.

Elsewhere, building on research in secondary grades that focused on how students explored speed situations to develop ideas about constant rates of change and the generalizations students developed from their understanding of speed (Lobato & Siebert, 2002), Ellis (2007a, 2011b) placed seventh-grade students in contexts in which they explored gear ratios in order to develop constant rates of change expressed as $y = mx$ and $y = mx + b$, for $m, b \in \mathbb{Q}$. Students' algebraic expressions and justifications were expressed as representations of their understanding of the ratio relationships between gear rotations and corresponding numbers of teeth.

As these studies suggest, quantitatively rich situations offer a useful context for generalizing and reasoning with quantities and the relationships between them. However, it is important to provide instructional settings that engage students in focusing on or generalizing relationships that are accurate, powerful, or even algebraically meaningful to them. Although the above examples make the case that quantitatively meaningful generalizations are possible, other studies at the upper elementary and middle grades levels, including classroom studies and design experiment research, demonstrate that these generalizations do not always occur. Much of the literature related to modeling, for instance, describes students' difficulties in making sense of realistic data (Lehrer & Schauble, 2004; Metz, 2004; Petrosino, 2003). Noble, Nemirovsky, Wright, and Tierney (2001) described a case in which the initial patterns developed by students were not helpful in their attempts to extend their reasoning; students noticed multiple patterns in the data they examined, but struggled to create algebraically useful generalizations. Van Reeuwijk and Wijers (1997) reported a simi-

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lar phenomenon in which students' initial perceptions were numeric in nature, and it was only with explicit support that they could develop generalizations of their understanding of relevant quantities and their relationships. A common phenomenon across these studies was students' attention to patterns and numeric relationships that did not emerge from a construction of quantities or quantitative relationships. These findings emphasized the fact that quantitative reasoning is not an activity that is automatically engendered by placing students in real-world or realistic situations. It is students' conceptions that create quantities, rather than the real-world contexts themselves.

The body of literature on students' generalizations within realistic contexts has thus identified both instances in which focusing on quantitative relationships encouraged the development of meaningful concepts and generalizations (e.g., Curcio, Nimerofsky, Perez, & Yaloz, 1997; Ellis, 2007a; Hall & Rubin, 1998; Lobato & Siebert, 2002; Slovin & Venenciano, 2008; Venenciano, Dougherty, & Slovin, 2012) and instances in which placing students in quantitatively rich situations did not guarantee that they would create algebraically useful generalizations (Noble et al., 2001; van Reeuwijk & Wijers, 1997). Although reasoning with quantities can support more sophisticated mathematical activity when students are themselves challenged to explore phenomena within problem situations, these studies suggest that students who fail to create new conceptual objects (such as ratio) built from quantitative relationships may not gain any additional benefit from being placed in quantitatively rich situations.

These findings about generalizing and reasoning with quantitative relationships suggest three important recommendations for introducing students in middle grades to functional situations through a quantitative reasoning perspective in a manner that can encourage mathematically productive generalizations (Ellis, 2011a):

1. Introduce functional relationships through quantitative situations that represent precise and reasonable, rather than approximate or contrived, data
2. Include quantitative situations with quantities that covary continuously rather than only discretely
3. Support sustained student attention to the quantities and their relationships in a given context or situation

Some problem situations involving contexts with messy or contrived data may interfere with students' sense-making abilities and may prevent students from directly manipulating quantities to form the necessary conceptual relationships for constructing initial ideas of linearity or other functional relationships. In contrast, contexts with exact but smoothly covarying quantities could afford the possibility of continuous covariational reasoning in a way that a discrete situation might hamper. Finally, because students' tendency to extract numbers and focus on pattern-seeking activities appears to be strong (Ellis, 2007b), it is essential that instruction also support students' engagement with quantitatively rich problems and attention to the quantitative referents of numbers and relationships.

*Representing Quantitative Relationships*

In the quantitative reasoning approaches advocated in elementary grades (e.g., Davydov, 1991; Schmittau, 2011), students reason about how to make unequal quantities equal (or equal quantities unequal) by combining or removing nonspecified amounts. As students compare nonspecified quantities, they can represent relational comparisons in mathematical equations. For instance, for mass $Y$ constituted by masses $A$ and $Q$ (see Figure 15.3), children may represent relationships with equations such as $Y = A + Q$ or $Y - A = Q$ (Dougherty & Slovin, 2004). Through this approach, children develop foundational ideas, such as the commutative property of addition, by combining attributes that have yet to be measured and representing their results symbolically as $A + B = B + A$.

A small number of researchers have enacted and studied the implementation of Davydov's elementary mathematics curriculum in school settings in the United States (Dougherty & Slovin, 2004; Schmittau, 2004; Slovin & Venenciano, 2008; Venenciano et al., 2012). Part of this research explored the different representations children used to express their thinking. In clinical interviews for the Measure Up project, researchers found that third-grade children were able to use both variable

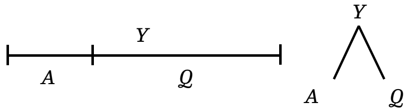


**FIGURE 15.3.** *Diagrams of relationships among parts of whole. From "Generalized Diagrams as a Tool for Young Children's Problem Solving," by B. Dougherty and H. Slovin, in M. J. Hoines and A. B. Fuglestad (Eds.),* Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education *(Vol. 2, p. 2-296), 2004, Bergen, Norway: Bergen University College.*

notation and generalized diagrams to represent situations and solve problems. Regardless of achievement level, students used multiple representations to represent the problems and associated actions. Students could solve problems with physical models, intermediate models such as line segment diagrams, and models represented through variable notation. In one study, Measure Up students were found to be better prepared for algebra than their non–Measure Up peers (Slovin & Venenciano, 2008; Venenciano et al., 2012). In addition, the Measure Up experience correlated positively with logical reasoning and algebra preparedness and correlated significantly with students' success in algebra (Venenciano et al., 2012).

Similar to the manner in which algebraic relationships are represented in Davydov-inspired approaches such as Measure Up, Ferrucci, Kaur, Carter, and Yeap (2008) described the "model method" as a route to algebraic thinking in which students make use of diagrams to represent quantities and relationships between and among quantities. Ferrucci et al. explored ways teachers can use this method to demonstrate how relationships emerge from concrete situations, even without the use of formal notation. This frees students to focus on organizing commonalities, viewing how changes in one quantity affect changes in another, and making grounded generalizations. For instance, Ferrucci and colleagues found that the model method enabled an elementary student without any knowledge of formal algebraic representations to solve a problem that would typically require the use of simultaneous equations:

> Mrs. Wu and Mr. Washington went to the Hillside Market to buy some fruit. Mrs. Wu bought 7 oranges and 4 apples for $4.80, and Mr. Washington bought 5 oranges and 2 apples for $3.00. What was the price of each fruit? (Ferrucci et al., 2008, p. 196)

Using the model method, students can use rectangles or portions of rectangles to represent unknowns. Ferrucci et al. (2008) demonstrated one approach in which students could use shaded rectangles to represent the price of each fruit and proceed to draw the model for the respective purchases (see Figure 15.4). Students can rely on such diagrams to look for relationships among the relevant quantities and to reason that if 5 oranges and 2 apples cost $3.00, then 10 oranges and 4 apples will cost $6.00 (Step 2). By comparing the diagrams in Step 1 with Step 2, they can then see that 3 extra oranges cost $6.00 – $4.80, or $1.20 more (Step 3). Thus if 3 oranges cost $1.20, each will cost

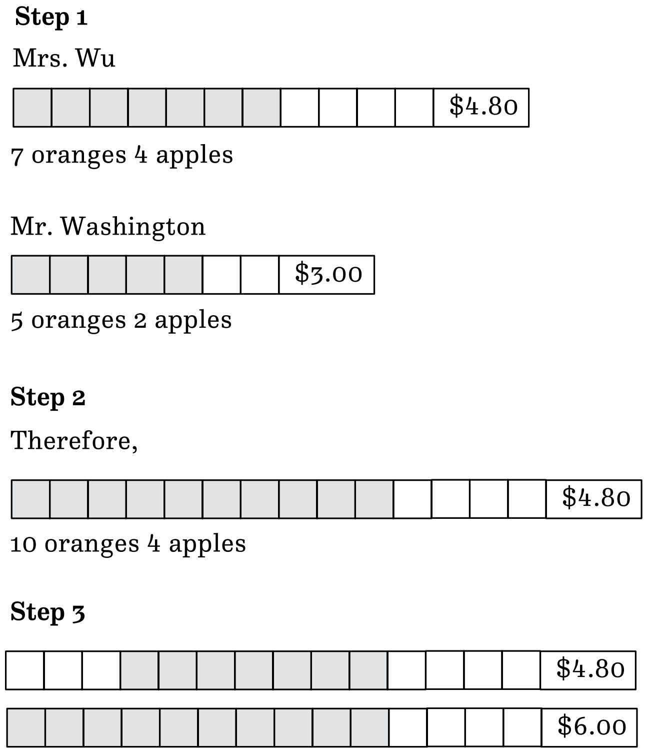


**FIGURE 15.4.** *Models illustrating a three-step solution. From "Using a Model Approach to Enhance Algebraic Thinking in the Elementary School Mathematics Classroom," by B. J. Ferrucci, B. Kaur, J. A. Carter, and B. Yeap, in C. E. Greenes and R. Rubenstein (Eds.),* Algebra and Algebraic Thinking in School Mathematics: Seventieth Yearbook of the National Council of Teachers of Mathematics *(p. 197), 2008, Reston, VA: National Council of Teachers of Mathematics.*

$0.40. The lower diagram in Step 3 depicts the total cost of the oranges as 10 × $0.40, or $4.00. Students can then determine that 4 apples cost $6.00 – $4.00, or $2.00, so each apple costs $2.00 ÷ 4, or $0.50. Similar to the model method, van den Heuvel-Panhuizen (2003) reported on the use of a bar model. A bar model is a strip on which different scales are depicted at the same time, illustrating how much of a quantity can be expressed through a different quantity. The bar model is used in a middle school percentage trajectory in which students learn about percentage, rational number, fractions, and decimals.

Modeling studies in upper elementary and middle grades have reported students' abilities to engage in multiple iterations of model development to identify and represent quantitative relationships such as ratios and proportions (Lehrer, Schauble, Carpenter, & Penner, 2000; Lesh & Harel, 2003; Lesh & Lehrer, 2003). For instance, Lehrer, Schauble, Strom, and Pligge (2001) reported on a study in which students developed an understanding of density as a constant ratio. Elsewhere, Lehrer et al. (2000) described students' focus on changing ratios to explore plant growth.

Gay and Jones (2008) noted that the first stage in the process of modeling is representing, which involves identifying the variables relevant to a situation and creating equations, expressions, diagrams such as the ones in Figure 15.3, tables of data, or other algebraic representations used to express relationships among quantities. Additional stages involve reasoning with the developed model to derive results and interpreting those results with respect to the original context. Tabach and Friedlander (2008) discussed how a context-based approach can facilitate the learning of four big algebraic ideas: (1) The role of variables and expressions as ways to represent meaningful phenomena of changes, (2) the difference between changing and constant quantities, (3) the lack of closure of algebraic expressions, and (4) the equivalence of such expressions. Modeling tasks that emerge from quantitatively rich situations in which students investigate and construct relations between quantities can motivate mathematical representations, not only in the form of diagrams or tables but also in terms of providing the context for algebraic expressions so that the power of algebraic notation can be exploited (J. Smith & Thompson, 2008).

Research with younger children's development of number and the modeling research with older children's attention to quantitative relationships suggests that supporting students' quantitative reasoning can foster a type of thinking—algebraic thinking—that is flexible and general in character. Although this thinking is not unique to quantitative reasoning, a focus on noticing and representing general relationships between quantities can support the type of conceptual development that will make algebra a sensible tool for thinking, because quantities, unlike numbers, are inherently indeterminate (Dougherty & Slovin, 2004; Slovin & Venenciano, 2008; Venenciano et al., 2012). Students can explore relationships between nonspecified or changing quantities, as described in Figure 15.3, in ways that can support the use of algebraic notation to represent those relationships. Much has been written about the difficulties encountered during the transition from arithmetic to algebra, but as Carraher, Schliemann, Brizuela, and Earnest (2006) emphasized, acceptance of these difficulties is tied to an impoverished view of elementary school mathematics in which generalization tasks are postponed until the onset of formal algebra instruction. Noticing and reasoning with quantities and their relationships can support the development of general mathematical ideas, ideas that with the proper instructional support students can represent with algebraic notation. In this sense, quantitative reasoning can provide a conceptual context for powerful forms of algebraic representation (J. Smith & Thompson, 2008).

Reasoning with quantities and their relationships can motivate an understanding of general relationships, which can in turn support a need for learning the notational tools of algebra. If students are developing mathematical ideas of sufficient complexity, the need to represent and reason with these ideas in, say, variable notation is strengthened (Ellis, 2007a; J. Smith & Thompson, 2008; Venenciano et al., 2012). Generally speaking, Kaput (1999) noted that students are more likely to generalize from their conceptions of meaningful situations and represent their conceptual activities based in those situations. Studies exploring how students reason with algebraic relationships in quantitatively rich situations have indeed shown that students can notice relationships and represent generalizations that hold quantitative meaning for them (Curcio et al., 1997; Ellis, 2007a, 2007b; Hall & Rubin, 1998; Lobato & Siebert, 2002; Noble et al., 2001; van Reeuwijk & Wijers, 1997).

### *Justifying Quantitative Relationships*

Although the activity of justifying is inherently mathematical, as we noted earlier, it takes on a special role in algebraic thinking when this activity occurs in the service of arguments about structure and relationships. Regarding developing justifications within quantitative reasoning, Morris (2009) suggested that representing and reasoning with representations of generalized quantities and their relationships could help younger children develop deductive arguments because the act of representing generalized quantities helps develop children's understanding that an action applies to an infinite set of objects, not just a single case.

Initial studies have demonstrated the promise of this line of inquiry for elementary and middle school students (e.g., Ellis, 2007a, 2007b; Hall & Rubin, 1998). Although few classroom-based research studies focused on quantitative reasoning have explicitly studied the role of justification, some do provide evidence of students' activities of justifying. Indeed, research has suggested that students in middle grades who focus on quantitative relationships in a problem situation are more apt to produce sophisticated justifications of their generalizations (i.e., use a “transformational proof scheme”; Harel & Sowder, 1998) than are students who focus on number patterns and procedures (Ellis, 2007a).

Hall and Rubin (1998) described an example from their work with fifth graders studying rate in the context of a ship’s speed. The authors found that students developed an ability to narrate rate as a relation between units of time and distance. In middle grades research, Ellis (2007a) studied students’ emerging understanding of linear functions within the contexts of rotating gears and same-speed situations and found that when the students situated their thinking within the quantitative context, they were able to produce justifications that were more powerful in their accuracy and deductive nature than those emerging from number-pattern reasoning alone.

Elsewhere, Lobato and Siebert (2002) detailed an account of middle school students providing rich explanations for the generalizations they produced, thus encouraging a quantitatively grounded justification for why a computer animated clown walking 2.5 cm in 1 second traveled at the same speed as a computer animated frog walking 10 cm in 4 seconds—a justification that was based on characterizing the relationship between the two composite units. Lobato and Siebert’s findings suggest that the activity of justifying about a general relationship provided an important context for reasoning proportionally with rates (Thompson & Thompson, 1994). The researchers were able to create an environment in which the students eventually produced justifications of their general statements that embodied the quantitative relationships important in isolating the attribute of constant speed.

## Summary of Findings and Directions for Future Research

At the 2001 ICMI Conference on the Future of the Teaching and Learning of Algebra, Kaput spoke about the “happy stories” that were beginning to emerge in the growing algebra research base (see also Lins & Kaput, 2004) in the elementary grades. This was not a naïve dismissal of the profound difficulties among adolescent-and-older students that had been documented through algebra research up to that point but a recognition that the focus on what students could *not* do necessarily obscured the part of the story that characterized what they *could* do. Since then, early algebra researchers have made significant progress in continuing to flesh out a research base that defines the cognitive foundations of children’s algebraic thinking. Carraher and Schliemann (2007) picked up this story, framed around the question “Can young students really deal with algebra?” (p. 675). They outlined an emerging research base that brought into relief the fact that the very difficulties adolescents experience with algebra might be better addressed through the development of children’s algebraic thinking in the elementary grades. They also pointed out that the same research showed that these difficulties would not be resolved through a more solid “grounding” in arithmetic as traditionally conceived. Our goal in this chapter has been to continue Carraher and Schliemann’s story by presenting more recent research that could particularly illuminate researchers’ understanding of how students engage in the core practices of algebraic thinking—generalizing, representing, justifying, and reasoning with mathematical structure and relationships. We briefly summarize the findings reported here and outline directions for future research.

We begin by noting that generalizing and reasoning with structure and relationships is based on a fundamental algebraic understanding of the symbols that capture how quantities are related (Kaput, 2008). The most common of these symbols—the equals sign—has been widely researched from kindergarten through grade 8, as researchers have sought to understand progressions in children’s thinking about the equals sign, the types of tasks that might promote a relational understanding of it, and how students’ understanding of it affects their algebraic work (e.g., Rittle-Johnson et al., 2011). What researchers now know is that children with a relational understanding of the equals sign can more successfully solve equations and produce equivalent quantities through substitutive thinking than children with operational misconceptions (e.g., Alibali et al., 2007; Jones & Pratt, 2012). Researchers now know that students’ lack of relational understanding might be attributed more to the curricular anomalies intrinsic to the study of arithmetic (McNeil et al., 2011) than to issues of developmental readiness. Furthermore, researchers know that instructional interventions can significantly shift students’ thinking from operational misconceptions to relational understanding (e.g., Blanton, Stephens, et al., 2015), thereby preparing students for the more formal study of algebra in middle grades.

Researchers now know that children as young as 5 and 6 years old have the potential to generalize and reason with structure and relationships through their explorations of operations on numbers in computational work (e.g., Russell et al., 2011a). Children can also make sense of relationships between quantities, whether these

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relationships are embedded in figural patterns or characterized through story contexts involving reasoning with nonfigural tools such as function tables (e.g., Blanton, Brizuela, et al., 2015; Rivera, 2010a). They can reason about generalized, continuous quantities as a way to construct the algebraic structure of the real number system prior to the introduction of number (Venenciano et al., 2012). Moreover, as children progress into the more formal mathematics of middle grades, they can begin to generalize and reason about patterns in rates of change grounded in physical contexts as they explore continuous quantities and relationships between them (e.g., Ellis, 2011b).

Studies cited here suggest that the struggles adolescents have faced with algebra might be *less* prevalent with younger students, in spite of the conventional wisdom that suggests such difficulties would be more pronounced in younger populations. This is particularly noticeable with variable notation. In spite of the well-known difficulties that adolescent-and-older students have with variable (MacGregor & Stacey, 1997)—students whose introduction to variable took place in traditional “arithmetic-then-algebra” settings—researchers now know that as early as first grade, children can successfully represent generalizations using variable notation in meaningful ways and develop relatively sophisticated understandings of such notation. Evidence also shows that some elementary grades students actually choose and are more successful with variable notation than with the representation more common to these grades—children’s natural language (Blanton, Stephens, et al., 2015; Brizuela et al., 2015; Carraher, Schliemann, & Schwartz, 2008; Dougherty, 2008; Isler et al., 2014). Those who advocate the introduction of variable notation in the early elementary grades suggest that variable notation acts as a mediating tool and that conceptual understandings and variable notation can fruitfully co-emerge (Brizuela et al., 2015). In our view, this renders the argument to withhold variable notation until students are “ready” less powerful. Indeed, as Carraher, Schliemann, and Schwartz (2008) observed, if we applied this argument to first language learning, “adults would never speak to newborns on the grounds that infants do not already know what the words mean!” (p. 237).

Finally, researchers now know that children in elementary and middle grades can learn to justify the generalizations they construct through arguments that reflect increasingly generalized forms, including representation-based arguments or arguments that use previously established generalizations as building blocks for a justification. And although students may rely heavily on empirical justifications initially, these arguments, too, may play an important but underappreciated role in helping students notice structure and form generalizations (A. J. Stylianides, 2007). A critical theme here is that supporting children in developing more generalized arguments needs to be an intentional focus of classroom instruction (Bieda, 2010; G. J. Stylianides et al., 2016, this volume), and even empirical arguments can bootstrap children’s justification in powerful ways if the classroom environment encourages the practice of justifying mathematical structure and relationships.

Questions that point to directions for future work certainly remain within K–grade 8 algebra research. To begin, more studies are needed to help researchers understand whether and why students respond more favorably to one mathematical context over another. For example, what are the affordances of tabular versus figural representations as elementary grades children generalize functional relationships? What are the affordances of a focus on quantitative relationships versus numerical patterns in the forms of justifications middle grades students are able to produce?

The studies reported here also raise an important question regarding the role of quantities in understanding functional relationships. That is, although attention to coordinated quantities seemed critical in the justifications middle grades students were able to produce, covariational (or coordinated) reasoning was not characteristic of elementary grades students’ thinking about functional relationships (e.g., Blanton, Brizuela, et al., 2015), even though elementary grades students could interpret the relationships they noticed in function tables in terms of the quantities involved. Part of this distinction might be explained by differences in how functional thinking is treated in these two grade bands relative to the increasing complexity of function concepts in middle grades, where issues regarding covariational reasoning and rates of change begin to emerge. To that end, another area that warrants researchers’ attention is the transition from elementary into middle grades. For example, how might generalizing functional relationships be better supported as students transition into more complex and formal mathematics in middle grades? What are the affordances of a covariational/coordinated approach versus a correspondence approach as students begin to explore concepts of slope in middle grades mathematics and how might the more common correspondence approaches found in elementary grades serve as a

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springboard into covariational thinking? Alternatively, what might a downward trajectory into elementary grades resemble for the specific genre of covariational/coordinated reasoning reflected in middle grades algebra research?

An additional point of transition that needs more research concerns how the number domains in which generalizing occurs shapes that activity. How does children's thinking about fundamentally different constructs (for example, continuous versus discrete quantities or messy versus exact data) affect how children reason about generalized quantities and their relationships? It is the case that much of early algebra research in elementary grades focuses on the use of whole numbers. As such, research that highlights the difficulties students have with non–whole number values in algebraic contexts (e.g., Christou & Vosniadou, 2012) raises the question of how sensitive students' algebraic thinking is to number domains that are more complex. How do findings change—if at all—when non–whole number domains are used and how might the use of non–whole number domains explain any difficulties middle grades students might have? Moreover, might it be the case that the difficulties middle grades students exhibit with functional thinking, for example, are born out of the complexity of considering relationships between continuous—as opposed to discrete—quantities? These same questions might be raised in the context of generalized arithmetic. That is, how does the complexity of students' algebraic reasoning with properties of arithmetic change when the domain being considered is, for example, fractions rather than whole numbers?

Another vital area of mathematical understanding, and a rich context for algebraic thinking, concerns the activity of justifying mathematical generalizations. Much of the research in this area has helped detail the nature of students' mathematical arguments. As noted earlier, however, research in elementary grades that has illustrated students' productive engagement with justifying generalizations often occurred in classroom environments with knowledgeable teachers or teacher-researchers who valued this practice. What is still needed across elementary and middle grades are studies that help identify the kinds of curricular and instructional support that will help teachers build environments that scaffold students' development of sophisticated arguments.

Moreover, because much of early algebra research takes place in typical classroom settings (or with populations from these settings), very little is known in regards to exceptional children. With algebra's historic gatekeeper effect, this population of students is particularly at risk, yet we know little about how such children think algebraically and how to support that. For example, how do children with learning differences engage with the algebraic thinking practices described here? How do they understand representations such as variable notation and what are the affordances of such representations over natural language? Is it possible that their differences might pose opportunities rather than challenges? If the goal of a K–grade 12 approach to teaching and learning algebra is to democratize access for all, such questions must be addressed.

Finally, as we look across the early algebra research base, we see an emerging shift in the types of methodologies needed for early algebra research that both reflects ways in which the field has matured and points to future directions. In particular, earlier studies over the last several decades necessarily focused on qualitative designs that could help the field flesh out conceptual terrains in children's algebraic thinking. Equipped with this knowledge, however, researchers are now in a position to explore other questions that get at the heart of early algebra's premise: a sustained, thoughtful development of children's algebraic thinking across elementary and middle grades can improve opportunities for students' success in school mathematics in secondary grades and beyond, thereby alleviating algebra's gatekeeper status. In this regard, quantitative studies that employ experimental designs to understand early algebra's impact on students' success in school mathematics are already underway (e.g., Blanton, Demers, Knuth, Stephens, & Stylianou, 2014; Blanton, Stephens, et al., 2015; Britt & Irwin, 2008; Brizuela, Martinez, & Cayton-Hodges, 2013; Schliemann, Carraher, & Brizuela, 2012) and represent an important future direction in early algebra research. Although there are certainly still questions to be answered regarding *how* children think algebraically (Carraher & Schliemann, 2007), impact studies of specific curricular interventions have the potential to provide concrete roadmaps for curriculum and instruction that can effect the goals of reforms around a K–grade 12 approach to school algebra instruction.

## Conclusion

The evidence suggesting that children *can* reason algebraically is perhaps even more compelling in the studies reported here than in previous syntheses of algebra research because of the increasing inclusion in recent years of very young populations of students—5- and

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6-year-olds—who are at the start of formal schooling. Such research, juxtaposed with Mason's (2008) argument that children bring nascent tendencies for generalizing to formal schooling that heretofore have not been adequately captured or exploited, continues to validate the reconceptualization of algebra as a K–grade 12 strand of thinking. The sometimes blurred boundaries between arithmetic and algebraic thinking, where children's implicit reasoning with properties of operations resides and which sets the stage for more formal noticing and reasoning with arithmetic relationships, underscores the futility of the historical arithmetic-then-algebra approach in which a "deep" understanding of arithmetic was expected to emerge prior to that of algebraic thinking. Research now provides a robust argument, honed through empirical research on children's thinking from the start of formal schooling and across diverse mathematical contexts, that children's algebraic thinking can (and should) emerge *with* their development of arithmetic thinking. Moreover, an overarching finding is that children's activity, particularly in the elementary grades, is fundamentally linked to and supported by their investigative work with problem contexts from which they can construct and derive meaning for the relationships they notice. Far from one-approach-fits-all, the diversity of research supports that there are multiple, fruitful entry points through which children can generalize, represent, justify, and reason with structure and relationships.

Our intent in this chapter was to be as inclusive as possible in the work presented here. No doubt, however, there are parts of the story that did not get told. For example, we chose to focus on students' algebraic thinking. This does not undermine the value of other important parts of the story, such as how curriculum and instruction can support the development of students' algebraic thinking (see Lloyd, Cai, & Tarr, 2016, this volume), or teachers' knowledge of algebra and their algebraic knowledge for teaching. Rather, it points to an expanding body of research—too extensive to be fully captured here—that can critically inform our international discourse on teaching and learning algebra and lead to real solutions to "America's algebra problem" (Kaput, 2008). That is certainly a happy story.

## Endnote

1. We take elementary grades or elementary school to mean kindergarten through grade 5, middle grades or middle school to mean grades 6–8, and secondary grades or high school to mean grades 9–12.

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