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Quantifying exponential growth: Three conceptual shifts in coordinating multiplicative and additive growth

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ABSTRACT

This article presents the results of a teaching experiment with middle school students who explored exponential growth by reasoning with the quantities height (y) and time (x) as they explored the growth of a plant. Three major conceptual shifts occurred during the course of the teaching experiment: (1) from repeated multiplication to initial coordination of multiplicative growth in y with additive growth in x; (2) from coordinating growth in y with growth in x to coordinated constant ratios (determining the ratio of $f(x_2)$ to $f(x_1)$ for corresponding intervals of time for $(x_2 - x_1) \geq 1$), and (3) from coordinated constant ratios to within-units coordination for corresponding intervals of time for $(x_2 - x_1) < 1$. Each of the three shifts is explored along with a discussion of the ways in which students' mathematical activity supported movement from one stage of understanding to the next. These findings suggest that emphasizing a coordination of multiplicative and additive growth for exponentiation may support students' abilities to flexibly move between the covariation and correspondence views of function.

1. Introduction: supporting student understanding of exponential growth

Exponential functions are an important topic both in school algebra and in higher mathematics. Not only do they play a critical role in college mathematics courses such as calculus, differential equations, and complex analysis (Weber, 2002), they also represent an important transition from middle school mathematics to the more complex ideas students encounter in high school mathematics. A focus on the conceptual underpinnings of exponential growth has increased in recent years; for instance, the Common Core State Standards in Mathematics (National Governor's Association Center for Best Practices, 2010) highlight the need to understand exponential functions in terms of one quantity changing at a constant percent rate per unit interval relative to another. Moreover, these ideas have also begun to appear at the middle school level in increasing degrees (e.g., Lappan, Fey, Fitzgerald, Friel, & Phillips, 2006).

This paper reports on the results of a teaching experiment with middle-school students who explored exponential growth as they explored the height of a plant growing over time. The analysis reported in this paper addressed the following research questions: (a) What conceptual stages can be identified in students' exploration of exponential growth as they investigated a scenario with continuously co-varying quantities; and (b) What factors contributed to students' conceptual change over

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time? The students entered the teaching experiment with a repeated multiplication understanding of exponentiation, and by the end of the teaching sessions they were able to coordinate the ratio of y_2 to y_1 for corresponding x-values. We endeavored to identify and characterize the stages through which students progressed as they shifted from a repeated multiplication understanding of exponentiation to a covariation understanding, in which students coordinated constant ratios for y with corresponding additive changes in x. In addition, we investigated the various factors, such as problem factors and instructional moves, which encouraged particular types of mathematical activity on the part of the students. In doing so, we identified and documented three conceptual shifts in students' understanding of exponentiation that we share in this paper, followed by a discussion of the ways in which students' mathematical activity supported these shifts. The findings suggest that a covariation approach to exponentiation, in which students are encouraged to coordinate the growth factor y_2/y_1 with corresponding additive changes in x ($x_2 - x_1$), can support a flexible conception of exponential growth that includes making sense of functions such as $y = ab^x$ for non-integer exponents.

1.1. The repeated multiplication approach and challenges in understanding exponential growth

A common textbook treatment for introducing exponential growth is the repeated multiplication approach. For instance, in the middle school curriculum Connected Mathematics Project (Lappan et al., 2006), students place coins called rubas on a chessboard in a doubling pattern, and then use tables, graphs, and equations to examine the relationship between the number of squares and the number of rubas. These types of tasks require students to perform repeated multiplication to solve a problem and then to connect that process to exponential notation. This approach follows the recommendations of researchers who suggest defining exponentiation as repeated multiplication with natural numbers (e.g., Goldin & Herscovics, 1991; Weber, 2002). However, generalizing to non-natural exponents may pose difficulties for students; for instance, an expression such as $2^{2/3}$ cannot be represented using repeated multiplication, and may be difficult to understand from a repeated-multiplication perspective (Davis, 2009).

Although the literature on students' and teachers' understanding of exponential growth is limited, there is research supporting Davis' concerns about the difficulties in generalizing one's understanding of exponentiation as repeated multiplication. For instance, Weber (2002) found that college students struggled to understand or explain the rules of exponentiation and could not connect them to rules for logarithms. Pre-service teachers have not fared much better; researchers have identified their struggles not only in understanding exponential functions, but also in recognizing growth as exponential in nature (Davis, 2009; Presmeg & Nenduardu, 2005). Although pre-service teachers appear to have a strong understanding of exponentiation as repeated multiplication, they experience difficulty in connecting this understanding to the closed-form equation and in appropriately generalizing rules such as the multiplication and power properties of exponents (Davis, 2009).

Research on middle school and high school students reveals difficulties as well; students struggle to transition from linear representations to exponential representations, or to identify what makes data exponential (Alagic & Palenz, 2006). In general, exponential growth appears to be challenging to represent for both students and teachers, and it is difficult for teachers both to anticipate where students might struggle in learning about exponential properties and to develop ideas for appropriate contexts that involve exponential growth (Davis, 2009; Weber, 2002). These documented challenges suggest a need to better understand how to foster students' learning about exponential growth, and for identifying more effective modes of instruction.

1.2. Alternate approaches to exponential growth

It is possible to conceive of exponential growth in other ways besides repeated multiplication: For instance, one can approach exponential growth as products of factors (Weber, 2002) or as what Confrey and Smith (1994,1995) refer to as a multiplicative rate constructed from multiplicative units. Weber (2002) offered a theoretical analysis of exponentiation relying on Dubinsky's (1991) Action, Process, Object, Schema (APOS) theory. Although this approach initially relies on an action understanding of exponentiation as repeated multiplication, Weber describes a scenario in which students would then transition to a process understanding by interiorizing the repeated multiplication action. Students would then view exponentiation as a function and be able to reason about its properties. This would enable students to consider expressions such as 2^3 as the product of three factors of 2, and ultimately students should be able to generalize this understanding to view a^b as b factors of a. Weber's analysis offers a vision for moving beyond the repeated multiplication view of exponentiation, but it remains an open question how students might undergo these conceptual transitions.

Confrey and Smith (1994,1995) introduced an operational basis for multiplication and division called splitting, postulating a different cognitive foundation for splitting versus counting. A splitting structure is a multiplicative structure in which multiplication and division are inverse operations, such as repeated doubling and repeated halving. Within this model, students also treat the product of a splitting action as the basis for its reapplication; thus, a split can be viewed as a multiplicative unit. Confrey and Smith (1994) suggested that basing multiplication on repeated addition neglects the development of a parallel but related idea of equal sharing, magnification, or repeated copies. They assert that "[b]uilding concepts of multiplicative rates constructed from multiplicative units should play a central role as students work on understanding how multiplicative worlds generate constant doubling times and constant half-lives" (p. 55).

Splitting as an operation can form the basis of what Confrey and Smith (1994) refer to as a rate-of-change approach to exponential functions; they found a number of such approaches adopted by students making sense of exponential situations.

For instance, students calculated additive rates of change and proportional new-to-old expressions of change, as well as making multiplicative comparisons. In the latter case, students would interpret a table with, for instance, a growth factor of 9 to be increasing by "a constant rate of nine." Confrey and Smith suggest that this comprises an important conception of multiplicative growth, which is found by calculating the ratios between successive *y*-values for constant unit changes in *x*-values. We highlight this conception as an important foundational idea for an approach to exponentiation in which students are encouraged to coordinate the growth factor (calculated as y_2/y_1) with corresponding additive changes in *x*, calculated as $(x_2 - x_1)$.

1.3. Covariation and quantitative reasoning

A popular approach to function relies on a correspondence view (Smith, 2003), in which a function is seen as the fixed relationship between the members of two sets. From this perspective, $y = f(x)$ represents *y* as a function of *x*, in which each value of *x* is associated with a unique value of *y* (Farenga & Ness, 2005). This static view underlies the typical treatment of functions in school mathematics. Thompson and colleagues (Thompson & Thompson, 1992; Thompson, 1994; Saldanha & Thompson, 1998; Thompson & Carlson, in press) and Smith and Confrey (Smith, 2003; Smith & Confrey, 1994) both discuss covariational reasoning, although in different ways. Smith and Confrey address how one can examine a function in terms of coordinated changes of *x*- and *y*-values:

> A covariation approach, on the other hand, entails being able to move operationally from y_m to y_{m+1} coordinating with movement from x_m to x_{m+1}. For tables, it involves the coordination of the variation in two or more columns as one moves down (or up) the table (Confrey & Smith, 1994, p. 33)

This conceptualization of covariation as the coordination of sequences is a natural fit for tasks employing tables that present successive states of variation. Students may operate covariationally with tables of data, as described by Confrey and Smith, even if their image of a quantitative situation is static rather than dynamic. From this perspective one must understand that quantities have a sequence of values.

Saldanha and Thompson (1998) extend the idea of covariation to consider the possible imagistic foundations that can support one's ability to think covariationally. They describe covariational thinking as a person holding in mind a sustained image of two quantities' values simultaneously. Castillo-Garsow (2013) similarly describes covariation as the imagining of two quantities changing together; students imagine how one variable changes while imagining changes in the other. In order to envision two quantities covarying, one must be able to envision each quantity varying (Thompson and Carlson, in press). In Saldanha and Thompson's description, a person thinking covariationally can couple two quantities in order to form a multiplicative object; once such an object is formed, one can track either quantity's value with the immediate and explicit understanding that at every moment, the other quantity also has a value.

Leveraging situations involving quantities that students can make sense of, manipulate, and investigate can foster their abilities to reason flexibly about dynamically changing events (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002). An approach that relies on imagining covarying quantities may be especially useful in helping students understand exponential growth, as this perspective is connected to how students think about contexts involving multiplicative relationships (Davis, 2009). Thompson (2008) argues that a defining characteristic of exponential functions is the notion that the rate at which the function changes with respect to its argument is proportional to the value of the function at that argument. Approaches emphasizing this concept could help students make strong connections between the change in *x*-values and the corresponding change in *y*-values, developing the understanding that the value of $f(x + \Delta x)/f(x)$ is dependent on Δx. Thompson (2008) proposed a way to think about exponential growth so that the argument and the function's value at that argument can vary smoothly by thinking of growth being linear over intervals whose endpoints are related exponentially. Although our approach differs, our goal was to help students coordinate (additive) changes in *x*-values with (multiplicative) changes in *y*-values to understand that, for a function such as $f(x) = 2^x$, the constant multiplicative growth for $\Delta x = 1$ would be 2, for $\Delta x = 2$ would be 2^2, or 4, for $\Delta x = 3$ would be 8, and for $\Delta x = 1/2$ would be $2^{1/2}$.

By quantities, we refer to conceptual entities (Thompson, 1994). In Thompson's theory, a quantity is a mental construction – a scheme – composed of one's conception of an object, a quality of the object, an appropriate unit or dimension, and a process for assigning a numerical value to the quality. Length, area, speed, and volume are all attributes of objects that can be quantified. Thompson (2011) highlights the importance of the dialectic between conceptualizing the attribute being measured and the ways in which it can be measured. He describes this dialectic as being at the heart of quantification – the process of "conceptualizing an object and an attribute of it so that the attribute has a unit of measure, and the attribute's measure entails a proportional relationship (linear, bi-linear, or multi-linear) with its unit" (p. 37). When students engage in quantitative reasoning, they operate with quantities and their relationships; quantitative operations are conceptual operations by which one conceives a new quantity in relation to one or more already-conceived quantities. For instance, one might compare quantities additively, by comparing how much taller one person is than another, or multiplicatively, by determining how many times taller one object is than another (Ellis, 2007; Thompson and Thompson, 1992; Thompson, 1994).

1.4. Continuous variation

In order to discuss continuous variation, it is helpful to first address the ideas of chunky and smooth reasoning (Castillo-Garsow, 2012, 2013; Castillo-Garsow, Johnson, & Moore, 2013; Thompson & Carlson, in press). A student who engages in chunky reasoning imagines that a change in y-values occurs in completed "chunks", for instance, after a certain amount of time has passed, such as one week. The student does not imagine that change occurs within the chunk unless she can re-conceptualize the change to a smaller chunk size, such as dividing a week into seven days, with each day having its own completed change. In contrast, when reasoning smoothly, a student imagines a quantity changing in the present tense. One can map from one's own current experiential time to a time period in the mathematical context without needing to resort to convenient units of time. Castillo-Garsow suggests that the continuous nature of smooth reasoning is a key ingredient for understanding the nature of exponential growth. This poses a significant challenge given that an exponential function is geometric; each term is found by multiplying the previous term by a growth factor. The sequential nature of exponential growth makes it natural to think about the next value, which is an inherently chunky way of thinking.

Thompson and Carlson (in press) further distinguish between *smooth continuous variation* and *chunky continuous variation*. Smooth continuous variation entails thinking about the variation of a quantity's value as its magnitude increases in bits while anticipating simultaneously that within each bit the value varies smoothly. Castillo-Garsow (2012) similarly describes continuous quantitative reasoning as a "repeated process of imagining the smooth change in progress of a quantity over an interval, followed by an actual or imagined numerical measurement of the quantity at the end of each interval" (p. 67). Chunky continuous variation, in contrast, entails thinking of the variation of a quantity's value as increasing by intervals of a fixed size. One can imagine a quantity's value shifting from a to b with the knowledge that there are values between a and b, but without imagining what happens in between. Chunky continuous variation differs from discrete variation, in which one can think of a quantity as taking specific values, changing from a to b, but without envisioning the quantity taking any value between a_i and $a_{(i+1)}$. Consistent with this approach, we endeavored to develop a didactic object (Thompson, 2002) that, to us, contained two continuously covarying quantities, along with means of support with the aim that middle school students would come to conceive the didactic object as we did. We anticipated that students may begin thinking about exponential growth in a discrete manner and hoped to support their means of shifting to continuous variation.

2. Methods

The study reported in this paper was part of a larger five-year project aimed at understanding middle school students' developing conceptions of functions through reasoning with quantities and their relationships. We first implemented an exploratory teaching experiment in order to (a) understand students' emerging conceptions of exponential growth, (b) identify shifts in students' conceptions over time, and (c) hypothesize potential mechanisms, both internal and didactical, which may have promoted the identified shifts. The results of this study then helped produce a hypothetical learning trajectory, which supported the design of an instructional sequence for a second, larger-scale teaching experiment. The results of the second teaching experiment led to the revision and refinement of the learning trajectory, which we then further refined through a series of classroom-based teaching experiments taught by a participating middle school teacher (see Ellis, Ozgur, Kulow, Dogan, et al., 2013; Ellis, Ozgur, Kulow, Williams, & Amidon, 2013). The findings in this paper are from the first teaching experiment, which constituted the initial phase of an ongoing, iterative series of studies aimed at understanding and fostering students' understanding of exponential growth.

2.1. Participants and the teaching experiment

We recruited five eighth-grade students (ages 13–14), accepted every student who volunteered, but restricted our analysis to the three students who participated regularly, Uditi, Jill, and Laura (all pseudonyms). Jill and Laura were enrolled in eighth-grade mathematics, and Uditi was enrolled in an eighth-grade pre-algebra course. None of the students had encountered exponential functions in their mathematics classes at the time of the study. The students participated in a 12-day teaching experiment (Cobb & Steffe, 1983; Steffe & Thompson, 2000) over the course of three weeks, in which the first author was the teacher–researcher. Two project members familiar with the teaching-experiment methodology observed and videotaped each teaching session, which lasted approximately 1 h. The project team met daily to debrief and discuss the events that transpired during the session. For the purposes of this paper, we present an analysis of the students' conceptual development over the course of the teaching experiment, with an emphasis on three shifts in understanding. The identification of these shifts occurred in post-hoc analysis after the completion of the teaching experiment.

Before embarking on the teaching experiment, the project team developed a tentative progression of tasks according to specific design principles, particularly focusing on the research on teaching covariation of quantities (Castillo-Garsow, 2013; Saldanha and Thompson, 1998; Smith, 2003; Smith & Confrey, 1994; Thompson and Thompson, 1992; Thompson, 1994) and continuous quantitative reasoning (Castillo-Garsow, 2012; Thompson, 2011) described in Section 1. However, the teaching experiment model demands a flexibility that requires the initial sequence of tasks to serve only as a rough model for instruction. Both during and between each teaching experiment session, we engaged in an iterative cycle of (a) teaching actions, (b) assessment and model building of students' thinking, and (c) revision of future tasks and invention of new tasks on an ongoing basis. In this manner we developed, during each teaching session, enhanced hypotheses of the students' understanding based on the previous cycle (Simon et al., 2010; Steffe & Thompson, 2000), and designed following tasks based on those hypotheses.

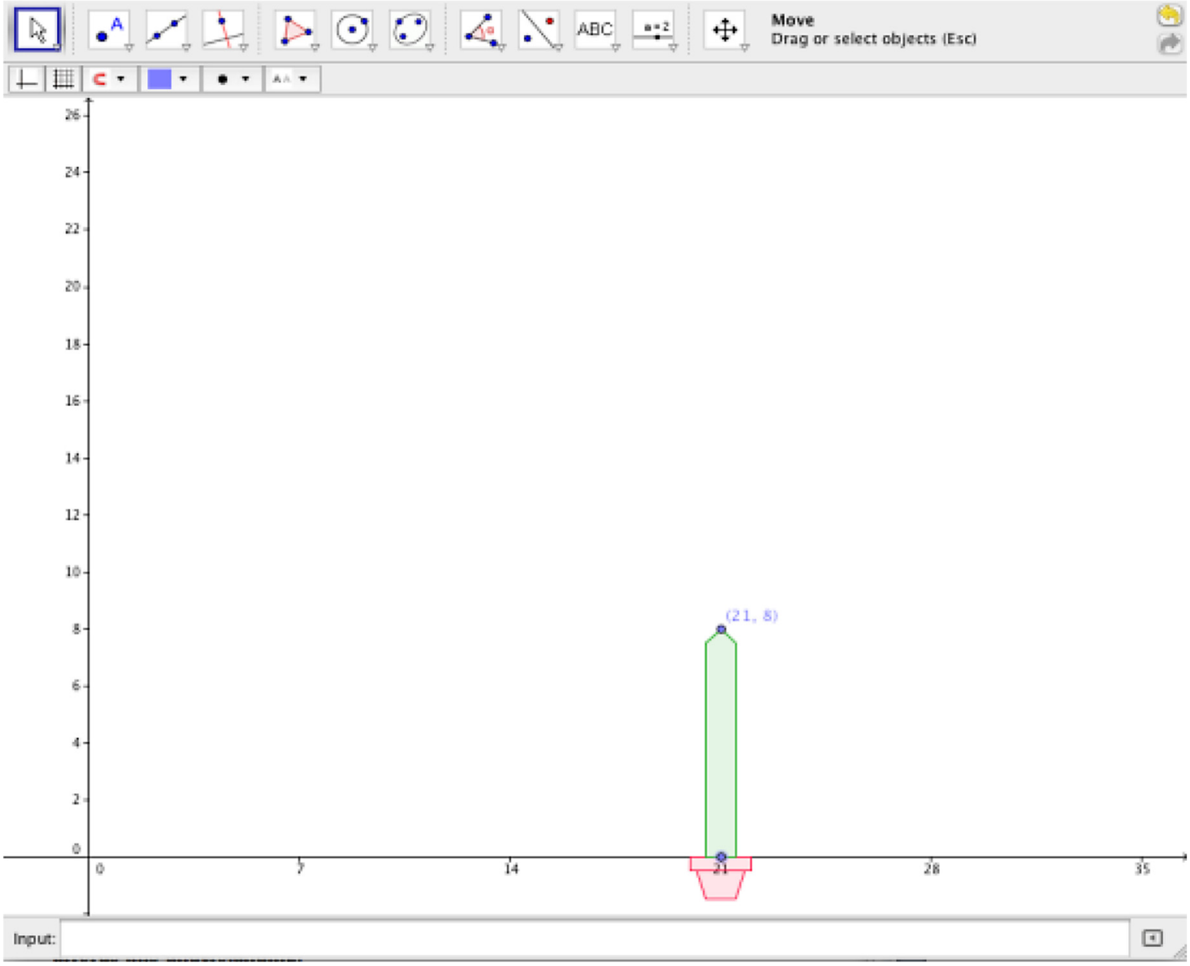


Fig. 1. The Jactus Geogebra script.

2.2. The Jactus context: reasoning with continuously co-varying quantities

Our aim was to introduce a context in which students could explore situations that, to us, entailed two continuously co-varying quantities. We aimed to design a setting that was understandable to middle school student students, with quantities that students could visualize, portrayed in a way that they could manipulate the quantities' values and observe the results of their manipulations. We developed a scenario in which a plant called the Jactus grew by doubling its initial height every week. Students could explore the growing Jactus plant by relating its height to a time at which it had that height. We did so via a specially designed Geogebra script (Fig. 1). Students could manipulate the image of the Jactus plant by dragging its base with the mouse. As they did so, the plant would continuously increase or decrease in height as students moved it along the time axis. The script also provided the numerical values for the Jactus' height for whatever value of elapsed time the students chose. Later in the teaching experiment, we changed the growth factor to values other than 2, and the initial height to values other than 1 in. Students had access to calculators throughout the teaching experiment. We considered the Jactus context "realizable" (Gravemeijer, 1994), in that it is a context that students can imagine and visualize. Although the Jactus context is not realistic in the sense that it could occur in real life, we found the tradeoff worthwhile in that it presents a situation with quantities that middle school students could understand as either varying in chunks or varying smoothly. It also presented a context in which students could manipulate the quantities and mathematize their values and the relationships among their values.

We must emphasize that placing students in situations that we conceive of as involving continuously varying quantities does not guarantee that students will engage in continuous reasoning. However, a context with covarying quantities that we see as varying continuously afforded us the opportunity to support students in seeing them likewise, and to observe difficulties they experience when reasoning about quantities discretely. It is possible to imagine and observe a plant that is somewhere between 1 in. tall when it starts growing and 2 in. tall after the first week. In contrast, it is more difficult to imagine a ruba coin that is in the middle of becoming two ruba coins.

2.3. Encouraging covariation

> We hypothesize that significant mathematical learning can take place as a result of students' engagement in appropriately selected and sequenced mathematical tasks. If individuals have the capacity to learn through their mathematical activity, the possibility exists to engineer a sequence of tasks that promotes the learning of individuals through their engagement with such a task sequence (Simon et al., 2010, p. 72).

Simon (2011) described a program of research aimed at understanding conceptual growth through studying the mechanisms of student learning as it occurs through mathematical activity when engaged in a sequence of tasks. Our participants entered the teaching experiment with an informal, repeated multiplication understanding of exponentiation. Building on that conception, our primary goal of instruction was to foster students' understanding of the following set of ideas for an exponential function $y = a \cdot b^x$ relative to changes in x. We designed an initial sequence of tasks to foster students' understanding of the following set of ideas:

1. The period of time x for the y-value to double (or increase by the growth factor b) is constant, regardless of the value of a or b.
2. There is a constant ratio change in y-values for each constant additive change in corresponding x-values.
3. The percentage growth in y is always the same for any same Δx; for instance, a Jactus plant will have the same percentage change in its height as it grows from Week 102 to Week 104 as when it grows from Week 2 to Week 4.
4. The value of $f(x + \Delta x)/f(x)$ is dependent on Δx.
5. The constant ratio change in y-values is dependent on both the growth factor b and on Δx in the following manner: $y_2/y_1 = b^{x_2 - x_1}$. This relationship will hold even when $\Delta x < 1$.

Our hope was that the students would eventually reach Idea #5, that $y_2/y_1 = b^{x_2 - x_1}$ even when $\Delta x < 1$, but we did not enter the teaching experiment with an expectation that this would occur. Indeed, only one student, Uditi, ultimately employed the relationship $y_2/y_1 = b^{x_2 - x_1}$ when $x_2 - x_1 < 1$. We relied on the above five ideas to guide the initial development of tasks, but also knew that the iterative nature of a teaching experiment would require an ongoing flexibility and revision of tasks.

In order to develop an initial task sequence to support the five ideas, we drew loosely on Carlson et al.'s covariation framework (2002). The framework addresses the reasoning involved in the representation and interpretation of graphical models in calculus, but served useful to guide our thinking about how to develop students' understanding of exponential growth. Carlson and colleagues described five mental actions, the first three being relevant to middle school students' reasoning with exponential functions (the fourth and fifth address average and instantaneous rates of change).

The first mental action is to *coordinate the dependence of one variable on another variable*. In the Jactus context, we developed activities to help students understand that the height of the Jactus depends on the amount of time that it had been growing. Students' interaction with the Geogebra scripts familiarized them with this dependence relationship. We then designed tasks in which students discussed the variables that could contribute to the Jactus' growth, drew pictures of the growing Jactus, and devised methods for keeping track of the plant's growth over time.

The second mental action is *coordinating the direction of change of one variable with changes in the other variable*. We wanted students to understand that as time increases, the plant grows taller, and as time decreases, the plant grows shorter. Activities with drawing pictures, interacting with the Geogebra script, and identifying relationships between how much time had passed and how tall the plant had grown encouraged this coordination. The third mental action is *coordinating the amount of change of one variable with changes in the other variable*. Students ought to develop the understanding that the growth in height is determined multiplicatively rather than additively; specifically, that new height values could be determined by multiplying the prior height value by the growth factor. Early activities aimed at this idea required students to create drawings depicting the plant's height at various time intervals. We combined these tasks with interpolation tasks, far prediction tasks, and comparison tasks across different plants with different growth rates and initial heights in order to encourage facility with coordinating change in height with change in time.

Additionally, we developed tasks that would encourage multiplicative comparisons of y-values with additive comparisons of x-values. We wanted students to understand that (a) the ratio of $f(x_2)$ to $f(x_1)$ will always be the same for any same $\Delta x = x_2 - x_1$, and (b) the value of $f(x + \Delta x)/f(x)$ is dependent on Δx. We designed tasks with non-uniform tables of data, including tables with varying gaps in time values, and asked students to determine missing entries and, later, find the growth factor. After students began to coordinate successive y-values by forming ratios, we introduced problems emphasizing different Δx values. We acknowledge that this approach emphasizes a focus on the values of the function at the ends of the intervals, rather than within intervals, but we considered this a reasonable conceptual goal for middle school students.

Our final goal was to enable students to coordinate the quotient $f(x_2)/f(x_1)$ and the value of the change in time $(x_2 - x_1)$ for values of Δx that were less than 1, ultimately being able to imagine this coordination for arbitrarily small increments in x. Weber (2002) argues that students should, in time, be able to generalize their understanding of repeated multiplication to make sense of what it means to be "the product of x factors of a" when x is not a positive integer. We suggest that rather than generalizing from repeated multiplication, students may have to construct a reciprocal relationship between roots and powers. This includes understanding that if a multiplicative change over an interval of time is b^n, then a change over m equally-sized subintervals of that interval must be b^k such that $(b^k)^m = b^n$; thus, k must equal n/m. For the Jactus context, this would mean understanding that if b is the plant's growth factor for a period of time, such as 1 week, then $b^{(1/a)}$ will be how many times taller the plant grows in $1/a$th of a week. Similarly, if a plant grows $b^{(1/a)}$ in $1/a$th of a week, then it must grow $b^{(1/a)a}$ in 1 week. The Jactus context could allow for a meaningful interpretation of this idea when students begin to contemplate how much the plant would grow in one day, for instance, given that it doubles in one week. We designed tasks in which students had to determine how tall a plant would be at non whole-number time values such as half a week or two and a half weeks. We then asked students to determine how much a plant would grow for many different time periods, such as for 10 weeks or for one day.

2.4. Data analysis

We employed retrospective analysis (Simon et al., 2010; Steffe & Thompson, 2000) in order to characterize students' changing conceptions throughout the course of the teaching experiment. We transcribed the entire teaching experiment and then produced a set of enhanced transcripts that included not only verbal utterances but also all images of student work, descriptions of relevant gestures, and other non-verbal student actions. We relied on the ideas outlined in Section 2.3 as a source of preliminary codes only, coding from the transcripts students' talk, gestures, actions, and written responses to tasks as evidence of understanding one or more of the concepts. For each student utterance or written task response, we determined as a group what the student's understanding appeared to be. This initial round of coding also produced emergent codes, which supported an emergent learning trajectory to account for the changes in student thinking that occurred during the teaching experiment. As an example, one student explained how she determined missing values in a table through repeated multiplication: "In one week it's going to be 1 inch and in 2 weeks it's going to be 2 inches, then in 3 weeks it's going to be 4 and in 4 weeks it's going to be 8." Examining this statement in concert with the student's written work and drawings, the research team determined that the student was explicitly connecting multiplication by 2 with a 1-week increase. In contrast, in prior days the student had described doubling actions without attending to time or the unit of increment for the weeks. Thus new codes emerged distinguishing between coordination of multiplicative growth with time not explicitly quantified and coordination of multiplicative growth in which time was explicitly quantified. The codes and the trajectory therefore evolved simultaneously in a cyclical manner through three full passes of the data set until the trajectory stabilized to reflect the final set of codes identifying students' conceptions. Although a presentation of the learning trajectory is beyond the scope of this paper (see Ellis, Ozgur, Kulow, Williams, et al., 2013), we present a portion of the trajectory accounting for three major shifts in students' understanding of exponential growth that emerged from this process of data analysis.

In the first round of coding, two members of the research team initially coded the entire data corpus independently. During this process they met weekly with the entire project team in order to discuss boundary cases and clarify and refine uncertain codes. Once this initial phase was complete, the full research team met weekly to code every transcript together, comparing each code and discussing any differences until reaching agreement. A subset of two researchers then re-coded the data set, again meeting weekly with the research team to discuss any final refinements. The iterative process of coding, refining, and recoding continued until no new codes emerged and no more refinement was necessary. We then chose 20% of the data corpus, making sure to include episodes encompassing the entire set of existing codes, for coding by a new project member for the purposes of inter-rater reliability. Inter-rater reliability revealed agreement at a level of 92%.

3. Results: from repeated multiplication to constant ratios to within-units coordination

In the following analysis we share three conceptual shifts we identified from the data (Fig. 2) that marked the students' transition from viewing exponential growth as repeated multiplication to coordinating the ratios of height values with the corresponding additive difference in time values. The three shifts are (1) from repeated multiplication to initial coordination of multiplicative growth in y with additive growth in x; (2) from coordinating growth in y with growth in x to coordinated constant ratios (in which students determine the ratio of $f(x_2)$ to $f(x_1)$ for corresponding intervals of time for $(x_2 - x_1) \geq 1$), and (3) from coordinated constant ratios to within-units coordination (in which students determine the ratio of $f(x_2)$ to $f(x_1)$ for corresponding intervals of time for $(x_2 - x_1) < 1$). In addition, to address research question (b), we also propose hypothesized mechanisms for each of the shifts, describing the ways in which students' mathematical activity with the sequence of tasks they encountered may have supported their ability to build more advanced ways of thinking. The structure of the results presents each of the shifts and subshifts in Fig. 2, followed by a discussion of the mechanisms driving the shifts.

After exploring the Geogebra script, the students identified repeated multiplication of the height as the mechanism determining the manner in which the Jactus grew. For instance, the students encountered a situation in which the Jactus was 1 in. tall when it began to grow, and it grew by quadrupling its height every week. Uditi described its growth this way: "They're all going up by like times 4, like 16 times 4 is 64 and then 64 times 4. . .then times 4, that's 1024." Explaining how she would determine the height of the plant at 7 weeks, Uditi stated, "Four times 4 is 16, 16 times 4 is 256 then 4. . .1024 times 4, 4096 times 4 and then it's 16,384." Absent from Uditi's language is an acknowledgment of the amount of time it took the Jactus to quadruple. The first shift we address is the one in which the students began to attend to corresponding time values when considering the height of the Jactus.

Conceptual Shift	Phases & Definitions	Day in which the Shift First Appeared
Shift 1: From repeated multiplication to initial coordination of growth in y with growth in x	1a: Students coordinate repeated multiplication for y with increases in x, but time is implicit and not quantified (e.g., the plant doubles "each time").	2
	1b: Repeated multiplication for y is coordinated with additive increases in x, with time explicit and quantified (e.g., the plant doubles every week).	2
Shift 2: From coordinating growth in y with growth in x to coordinated constant ratios	2: Students begin to explicitly coordinate the quotient $f(x_2) / f(x_1)$ with corresponding intervals of time $(x_2 - x_1) \geq 1$.	5
Shift 3: From coordinated constant ratios to within-units coordination.	3a: Re-unitizing: Students construct new units of change in x	9
	3b: Developing a scaling image of multiplicative growth	6
	3c: Coordinating the quotient $f(x_2) / f(x_1)$ with corresponding intervals of time $(x_2 - x_1) < 1$.	10

Fig. 2. Three conceptual shifts in students' understanding of exponential growth.

3.1. From repeated multiplication to initial coordination of growth in y with growth in x

3.1.1. Time is not quantified

On Day 2 of the teaching experiment the students encountered a far prediction problem: "Given a Jactus with an initial height of 1 inch that doubled every week, how tall would it be at 30 weeks?" Laura's response indicated an understanding of repeated doubling, but there was no evidence of her attending to weeks as a quantity:

> You can do, like, so here's 8 (inches), and then the next is 16 (inches) and I guess like
> I said yesterday, it goes up, if that's the rate. . .for 32 (inches) you would do 32 times 2, and then you have the result for the week, for week 6 and then you just keep going.

Although Laura referenced "Week 6", she appeared to coordinate how the height grew with the weeks only in the sense that the weeks served as a way to keep track of how many times to double. It appeared that Laura viewed each week as a marker or a counter rather than a quantity of time.

The next day the students explored the same growing Jactus given several ordered pairs: (0,1), (1,2), (2,4), (5,32), and (9, 512). Uditi explained how she determined the plant's height for the missing weeks 3 and 4:

> So it's like times 2 is 2, 1 times 2 is 2 (inches) and then 2 times 2 is 4 (inches). So then for the missing things 4 times 2 is 8 (inches), so for 3 (weeks) it's 8 and then 8 times 2 is 16 (inches) and then 16 times 2, 32 (inches) over there. So that's. . .so it's times. . .so you have to do times 2 to find the next one, so I did times 2 and found the missing ones.

Uditi identified a multiplicative pattern in the inches, extending that pattern to fill in the missing heights for Weeks 3 and 4. Similarly to Laura's reasoning, the weeks fulfilled a counter role, with each week serving as a tick mark to help her keep track of how many times she had to double the height.

3.1.2. Time becomes quantified

In an attempt to encourage coordination of the plant's height with the number of weeks it had been growing, the teacher–researcher introduced a task with a picture of a 1-in. tall doubling Jactus at 0 weeks that asked students to draw the plant's height after 1 week and after 3 weeks. Although Jill's picture included the correct number of inches, her drawings were not to scale (Fig. 3):

Jill's drawing indicated the beginning of a coordination of the plant's height with the number of weeks in that she included both height and weeks as labels for each plant. However, the fact that the Jactus at 3 weeks is only twice as tall as the Jactus at 1 week suggests that Jill did not explicitly attend to the number of weeks as a quantity: She did not conceive of the gap in time between the plant at 1 week and the plant at 3 weeks as a 2-week gap in which the plant would double twice. Uditi and Laura produced similar drawings. The teacher–researcher (TR) then asked the students to think about Week 4:

TR:	So what would happen the next week and Week 4?
Uditi:	It would be more bigger.
TR:	It would be more bigger?
Jill:	It would be double, it's. . .

TR:	Oh, it would be double? Double what?
Jill:	Well, the next week would be double the last week.
TR:	Oh, the next week would double the last week? Okay.
Jill:	Like Week 4 would be 16 inches.
TR:	So can you draw what that looks like?

After this conversation, the students could draw a plant at Week 4 that accurately depicted its height as twice as tall as it had been for Week 3. The students became more explicit about a plant doubling from Week n to Week $n+1$, but could not coordinate this doubling action with multiple-week jumps. In addition, their images of growth did not entail growth between identified moments in time.

Over the next two days the students encountered more tasks requiring them to determine the plant's height for skipped or missing weeks; the intention was to highlight the need to explicitly attend to elapsed time in weeks as well as the height in inches. The students demonstrated this attention to both quantities in their work, but did not yet coordinate the multiplicative growth in height with multiple-week gaps. For instance, working with a table of values in which one had to determine the plant's height for Week 10 and Week 15 (Fig. 4), Laura coordinated her action of doubling the inches for each successive week by filling in the gaps in the table. Her language reflected an explicit attention to both weeks and inches: "For 4 (weeks) I got 48 (inches)." Laura could only double the previous week's height to find the next week's height, not yet understanding that this process could be truncated; for instance, that she could find the Jactus' height at Week 10 by taking its height at Week 8 and multiplying it by 4, or 2^2 for a jump of 2 weeks. Thus it was necessary for Laura to include every missing value in the table in order to accommodate missing weeks.

3.1.3. Student activity fostering the coordination of growth in y with growth in x

Uditi, Laura, and Jill encountered two types of tasks encouraging the coordination of repeated multiplication in height with additive changes in time. The first was picture-drawing tasks similar to the one in Fig. 3. Students were asked to draw pictures depicting the height of the Jactus first for successive weeks (such as Week 2 through Week 4), and then for skipped weeks. By introducing missing weeks, the tasks encouraged students to mentally coordinate the skipped week's height with the skipped week's time, holding those two quantities simultaneously to then again repeat the multiplication action coordinated with an additional week in time. We suggest that the action of skipping one or more weeks helped the students connect the action of imagined doubling with an imagined one-week increase.

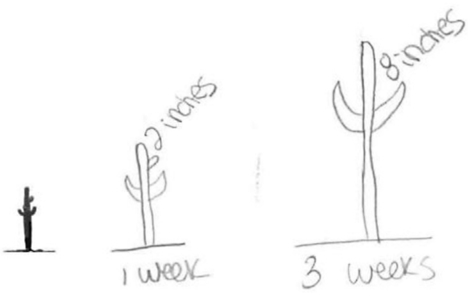


Fig. 3. Jill's drawing of the growing Jactus.

Weeks	Inches
0	3
1	6
2	12
3	24
4	48
5	96
6	192
7	384
8	768
9	1536
10	3072

Fig. 4. Laura's table of values for the doubling Jactus.

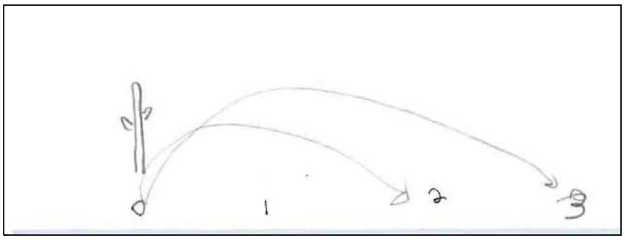


Fig. 5. Teacher–researcher's ad hoc task.

The second type of task was the presentation of non-uniform tables such as the one shown in Fig. 4. The task sequence began with missing height values, then shifted to include missing week values, and finally introduced far prediction questions. The students engaged in the same doubling or tripling actions as they completed each new table, and could generalize from common features across the tables. Initially, all three students needed to include every missed week in order to keep track of their repeated multiplication actions on a week-by-week basis. When it became clear that the students would need explicit support coordinating multiplicative change with additive change for multi-week gaps, we hypothesized that introducing new tasks with larger gaps between the weeks could encourage students to think about how many times they must multiply the height by the growth factor for a given number of weeks, particularly if we highlighted an attention to the number of missing weeks in discussions with the students.

3.2. From initial coordination of growth in y with growth in x to constant coordinated ratios

On the third day the students encountered three data sets, in table form, of the following ordered pairs of time and height: [(2, 16), (3, 64), and (5, 1024)]; [(4, 256), (8, 65,536), and (9, 262,144)]; and [(1, 4), (6, 4096), and (7, 16,384)]. The students had to determine which of the data sets represented the fastest-growing plant, or whether they all grew at the same rate. (All of the ordered pairs are for the function $y = 4^x$, although the students did not have that information.) Uditi noticed that in the first data set, the ordered pairs (2, 16) and (3, 64) were consecutive points and 4 times 16 in. was 64 in. She therefore hypothesized that the growth factor for the first data set was four. Uditi's work then showed her first attempt to coordinate multiplicative growth in height with a change in time of two weeks. First, she multiplied 64 in. by 4, to get 256 in., and then she repeated the multiplication action to get 1024 in. for 5 weeks. Note that at this stage Uditi appeared to conceive of multiplication as iteration; she imagined 64 in. as the value of the plant's height, which she could iterate 4 times to produce a plant 4 times as tall, or 256 in., at 5 weeks. She could then imagine repeating that action to find the plant's height at the next time period, 6 weeks. However, it is unlikely that Uditi had an image for the growth in height between the weeks, because she then re-did the problem by multiplying 64 in. by 8 to account for the two-week gap. Finding the answer of 512 in., Uditi realized the action of multiplying by 8 was incorrect because it did not match the height she expected at 6 weeks, 1024 in.

The teacher–researcher discerned that Uditi was attempting to coordinate multiplying by the growth factor of 4 twice with the two-week additive change in time. Multiplying by 8 rather than by 4×4, or 16, revealed Uditi's struggle to correctly consolidate the action of multiplying by 4 twice. The teacher-researcher drew a picture of the Jactus at Week 0 (Fig. 5) and then asked the students to anticipate what they would have to calculate in order to jump directly from Week 0 to Week 2, and from Week 0 to Week 3.

TR:	Okay, what do you have to do to get from Week 0 to Week 2 for the height?
Uditi:	One times 16, so like, from this (gestures to Week 0) to this (gestures to Week 1) it's times 4 so 4 times 4 is 16.
TR:	Oh, okay, so times 4 and times 4 or times 16?
Uditi:	Uh huh.
TR:	Okay. I'm going to write it as times 4 and times 4. What do you have to do to get from Week 0 to Week 3?
Uditi:	Times 4 times 4 times 4. (TR draws an arrow from Week 0 to Week 3 with $4 \times 4 \times 4$ written next to it.)
TR:	Okay and what about from Week 0 to Week 4?
Uditi:	You have to add it four times.
TR:	Okay. Would this help you figure out how to do a big number directly, like say, Week 20?
Uditi:	Yeah.
TR:	How would you do it?
Uditi:	So like you add 20 more times. . .like you add 20 more times four like that.
TR:	Oh, okay. So that's kind of a big number so let me make a smaller number since 20 would be a whole lot. What if we were just jumping directly to, say, 7, then how would you do it?
Uditi:	[Multiplying on her calculator]: Four times 4 is 16, 16 times 4 is 64, 64 times 4 is 256 then four. . .1024 times 4, 4096, times 4 and then it's 16,384.

Uditi's language of "add 20 more times" belied a possible conflation of repeating the process of multiplying by 20 factors of 4 (or 4^{20}) and multiplying by 4×20. In order to clarify which action Uditi meant, the teacher–researcher asked her to determine the height after 7 weeks, a process easier to calculate. Uditi's response was to multiply the plant's height by seven

Weeks	Inches
0	0.1
8	25.6
18	26,214.4
30	107,374,182.4
13	______
x	?

Fig. 6. Non-uniform table of exponential data.

factors of 4 to account for the 7-week gap in time. This represented an early coordination of multiplicative growth in *y* with additive growth in *x*. Uditi needed to explicitly account for each step between the 7-week gap, coordinating an increase in 1 week with the action of multiplying by 4. We hypothesize that Uditi continued to conceive of multiplication as iteration, imagining a process in which a height of 4 in. became 16 in. by repeating the height four times, or stacking four identical copies of the 4-in. plant end to end to create a 16-in. plant. Note that this image is static and does not provide a way for Uditi to think about how the plant would grow between a height of 4 in. and a height of 8 in.

By Day 5 the students were confident in an initial coordination of multiplicative growth in *y* with additive growth in *x* when working with non-uniform data tables (Fig. 6). The function for these data is $f(x) = 0.1(2^x)$, but the students did not have that information when they encountered the table. Laura was absent on this day, and her attendance grew sporadic for the remainder of the sessions. Jill's response demonstrated a recognition of the need to coordinate the change in the Jactus' height with the change in weeks for multiple-weeks spans. She knew that she would have to multiply the height of the Jactus, 0.1 in., by the growth factor 8 times in order to account for the 8-week gap. Jill guessed and checked with various growth factors, starting with 5, and then 3, and then finally 2. She checked her work by creating a new table with Weeks 2, 3, 4, 5, 6, and 7, so that she could systematically multiply by 2 each time, concluding, "They are increasing by 2 each week." Uditi employed a similar strategy, creating a new table so that she could fill in the missing weeks.

The students' need to fill in the missing values of the table indicated a difficulty with consolidating this process for multiple week gaps; for instance, multiplying directly by 2^8 rather than multiplying by eight factors of 2. During the teaching experiment, the project team members hypothesized that providing students with problems containing only two data points relatively close together might encourage the beginning of this consolidation. For instance, given two points (14, 956593.8) and (16, 8609344.2), the quotient of the two height values is 9. The research team members predicted that students would recognize small perfect squares such as 9, 16, and 25 and understand them as 3^2, 4^2, or 5^2.

Both students were indeed able to recognize perfect squares for two-week gaps. For instance, given the points (0, 0.1) and (2, 1.6) both students divided the heights 1.6 in. by 0.1 in. to find that the plant had grown 16 times as large in two weeks. Uditi explained that the growth factor must be 4, saying "I tried to multiply and then multiply it again, the same number." This suggests that Uditi understood the quotient, 16, as a representation of how many times taller the plant's height was at 2 weeks than at 0 weeks. She also likely understood that the plant was therefore 4 times taller at 1 week than it was at 0 weeks, because she did not need to create a new table or fill in the height for the missing week. Similarly, for the problem above with the points (14, 956593.8) and (16, 8609344.2), Jill explained, "I tried 3 times 3 and it was the 16 week number, and so then I figured out if I did the 14 week number times 9 it would give me this." Jill too could now make this leap without filling in the missing value.

Both Jill and Uditi were beginning to recognize that the power of a number is the result of repeated multiplication; for instance, 3^2, or 9, is the result of multiplying by 3 twice. Further, we hypothesize that both students had begun to project their actions of repeated multiplication to a reflected level where they could imagine the result of multiplying repeatedly and use this image in place of actually performing the repeated multiplication. It is likely that their ability to imagine such a result was nascent, as they could not yet generalize their reasoning to any gap beyond two weeks; for instance, Uditi was unable to explain that the plant would grow $4 \times 4 \times 4$ times, or 4^3 times, between week 2 and week 5 because there was a 3-week gap.

3.2.1. Reliance on images of repeated multiplication

The students' reflection on their actions of repeated multiplication helped to solidify the connection between the number of weeks that elapsed between two heights, the number of times one would have to multiply the plant's height by its weekly growth factor, and the power of the weekly growth factor that related the Jactus' initial and final heights. In time this connection became explicit. For instance, when the students encountered two data points with a 5-week gap (the Jactus was 3,355.443.2 in. tall on week 24, and it was 107,374,182.4 in. tall on week 29), they were able to determine the unknown growth factor. Jill took the ratio of the two height values and found it to be 32, and then wrote, "_ _ · _ _ · _ _ · _ _ · _ _ = 32", searching for a number for which 32 is the 5th power. Thus Jill was able to anticipate that 32 had to be the fifth power of some growth factor.

The students' struggle to explain and generalize their thinking at this stage suggests the nascent nature of their ability to connect multiplicative growth in height with additive growth in time for multiple-week gaps. The teacher–researcher asked Uditi to describe generally how she would determine the growth factor of a plant given only two height values any number of weeks apart. Uditi could use a specific example to think through her actions: "So, if they are like 5 weeks apart then you would try to find a number that times itself by 5. . .and then, so. . .you have to try to find the difference." By "the difference" Uditi meant the weekly growth factor, and again her language suggests an image of multiplication as repeated iteration. Jill remarked that it would be helpful to have what she called a "shortcut", and at this point the teacher–researcher introduced the notation b^m as a way to express m factors of b. Although the students quickly made use of this representation, repeated multiplication as iteration remained a strong focus for their actions and inscriptions. Uditi and Jill could coordinate the multiplicative growth in height for small-week gaps, gaps for which they could imagine iterating by the growth factor four or five times. However, they had difficulty generalizing their thinking to a gap that was arbitrarily large or small. In addition, their coordination of multiplicative growth in height with time was discrete; they could think of time as taking on specific values, such as 1 week or 2 weeks, but did not think about how time varied between those values. Instead, time jumped from one value to the next, and height jumped along with it.

3.2.2. *Student activity fostering constant coordinated ratios*

The process of retrospective analysis led to an identification of four types of student activity that supported a coordination of the multiplicative growth in height with the additive growth in time for small-week gaps. First, tasks presenting multiple data tables for the same Jactus encouraged the students to compare their actions of repeated multiplication across the different tables. Further, they had to anticipate that determining the plant's height for a jump of two or three weeks would require iterating the height by the plant's growth factor two or three times. Second, directing students' attention to jumps across two and three weeks, such as that seen in the conversation about the drawing in Fig. 5, oriented them to attend explicitly to both a gap in weeks and to the number of weeks constituting the gap. Asking students to anticipate actions they would take before doing any calculations further served to draw their attention to the connection between repeatedly multiplying the height and repeatedly adding the weeks.

Third, providing tasks with two data points with only a two- or three-week gap enabled the students to recognize known square or cube numbers. Reflecting on these repeated tasks enabled students to anticipate that once they found the quotient of the heights for an m-week gap, they would need to determine what number they would need m factors of in order to return the quotient. The following conversation between Uditi and the teacher–researcher indicates evidence of Uditi's ability to *imagine* the Jactus multiplying its height by 4 three times, without having to actually fill in the missing values in the table:

TR:	One thing I'm not sure of is exactly why it works, like why does dividing by the two to get a number, in this case 64, and then saying something times something times something equals 64 give you the multiplier factor, why does that work?
Uditi:	Because like this number [the first height value] times 64 equals this number [the second height value], so the difference [in weeks elapsed] is 3 so it can be like, that times that times that because there are some numbers in the middle too.

Uditi recognized that the table had missing values, but she no longer needed to physically fill in those missing values. Instead Uditi could replace the physical act of coordinating multiplication with addition by keeping written track of the missing weeks with mental coordination.

Finally, the students encountered explicit reflection tasks, such as, "How do you figure out how the Jactus is growing if you are given any two points in a table?" These prompts encouraged students to provide general descriptions. As seen in Uditi's reliance on a generic example, it was challenging for the students to describe their process in general terms. We posit that the students' attempts to articulate a general process encouraged reflection on what had been common across their actions with multiple problems.

3.3. *From coordinated constant ratios to within-units coordination*

The students' reliance on images of repeated multiplication made it difficult to coordinate multiplicative growth in height with growth in time for increases that were very large, arbitrarily small, or not whole-number gaps. For instance, Uditi encountered a table of data in which she had to determine whether the week and height values represented exponential growth. Uditi wanted to check whether she could find a growth factor that explained the increase in height for an 8-week gap as well as the increase in height for a 0.4-week gap (Fig. 7).

Uditi was able to guess and check (by repeatedly multiplying different potential growth factors) that a growth factor of 3 would account for the increase in height between Week 5 and Week 13 (it grew 6561, or 3^8, times as tall). However, when Uditi attempted to make sense of the 0.4-week gap between Week 0.1 and Week ½, she was unable to correctly represent the plant's growth. Uditi knew that it grew approximately 1.55 times as large, but because her image for growth was that of repeated multiplication, she could not imagine a 1.55 multiplicative increase occurring for an increase of 0.4 weeks. Instead, Uditi represented 1.55 as some unknown value used as a factor four times (see Fig. 7) and thus became stuck. This was likely

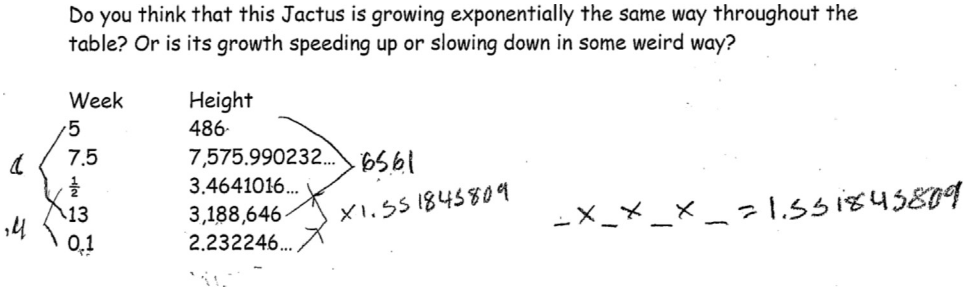


Fig. 7. Uditi's work on a table of non-uniform values.

complicated by the fact that Uditi did not have access to conventional mathematical tools such as fourth roots, so she did not know that taking the fourth root of 1.55 would give her the 0.1-week growth factor.

The shift to coordinating multiplicative growth in height with any time increase was one that encompassed several interim shifts in thinking. In order to make sense of growth for non-standard time increases, the students had to rely on images other than repeated multiplication to imagine multiplicative growth in height. This transition required students to first engage in a process of *re-unitizing* in order to make sense of large chunks of time, and then imagine growth in the plant's height as a continuous stretching or scaling process that could be expressed algebraically as b^x. We discuss each of these interim shifts in turn.

3.3.1. Re-unitizing

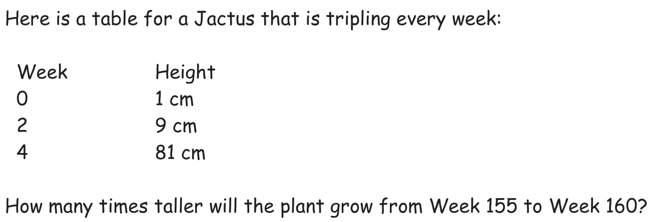
One strategy that emerged as students began to grapple with larger time chunks was the process of re-unitizing, or taking a new chunk of time (such as two weeks, three weeks, or another time span) as a unit, and operating on that unit as the basis for coordinating multiplicative growth. A rudimentary version of this can be seen with Jill's work on a task requiring students to determine growth for a two-week chunk of time (Fig. 8).

One goal of the task was to determine whether students understood Idea 3 from Section 2.3, that the ratio of the growth in y is always the same for any same Δx, so if the plant grows 243 times as tall from Week 0 to Week 5, it will also grow 243 times as tall from Week 155 to Week 160. Because the numbers are so large for large week values, the students could not make any direct calculations to determine how many times taller the plant would grow. Jill incorrectly determined that the plant would grow 21 times taller by creating a unit of 2 weeks. She explained, "Every two weeks it goes up by 9, 9 times 2 is 18 and then there's like 1 (week) left over and so then that's only 3. . .and so it'd be 21 times." Jill appeared to think of two weeks as unit of time, so that she could think about the plant growing 9 times as tall for each two-week chunk. However, Jill then shifted to additive reasoning by concluding that the plant would grow 18, rather than 81, times as large in four weeks.

Uditi demonstrated re-unitizing in a similar aim to make sense of a task that asked students to compare how a doubling Jactus grew every week, every three weeks, and every ten weeks. All three students knew that a doubling Jactus grew twice as tall each week, and produced the equation $y = 1 \cdot 2^x$ to express that relationship. Uditi then produced a table with the points (3, 8), (6, 64), and (9, 512) to conclude, "It grows up by times 8 every 3 weeks." As evidence that she conceived of the three-week chunk as a new unit of time, Uditi used the "8 times as large for every 3 weeks" as a basis for operating, explaining, "It's 8 for 3 weeks, 8 times 8. . .I mean, 8 times 8, yeah, equals 64 for the 6 weeks." Thus Uditi could coordinate multiplicative growth by 8, a new growth factor, for a chunk of 3 weeks in time, a new time unit. Both Uditi and Jill were confident at this stage that the Jactus would also grow 8 times as large from Week 100 to Week 103, without having to calculate. Uditi explained, "Every number is going by eight"; although her language was imprecise, Uditi's use of "every number" further indicates that a 3-week span of time was a new unit for which the plant grew 8 times as tall.

3.3.2. Student activity fostering reunitizing

Tasks presenting data in well-ordered tables with time increases other than 1 week may have supported students' abilities to take other time spans, such as 2 weeks or 3 weeks, as the standard unit of time. The students encountered multiple tables such as the one in Fig. 8 and were asked to think about how the plant would grow for different time increments. After


Here is a table for a Jactus that is tripling every week:

Week	Height
0	1 cm
2	9 cm
4	81 cm

How many times taller will the plant grow from Week 155 to Week 160?

Fig. 8. Task requiring students to determine growth for a five-week chunk.

encountering such tables, the students then began, unprompted, to create their own tables of data with time increments other than 1 week. Repeated activity of increasing the time by a new unit, such as 3 weeks, may have helped solidify the connection between multiplicative growth (such as, it grows 8 times as tall) and additive growth (for each 3-week span) for the new time span.

The students constructed a new kind of unit, one that is more than a ratio and more than an increment of time. Rather, the new unit is a joining of multiplicative growth with additive growth. For instance, with the task in Fig. 8, Uditi, unlike Jill, could correctly operate with the [+2 weeks: ×9] composite unit, repeating it once to get [+4 weeks: ×81], and then adding on a "half" [+1 week: × 3] to conclude that the final composite unit of increase would be [+5 weeks: ×81 × 3], or [+5 weeks: ×243]. We believe Uditi could envision the Jactus growing 9 times as tall in a 2-week span, and that she could also imagine the plant growing 3 times as tall in 1 week, and then 3 times as tall again in the second week. Thus Uditi could now imagine one time point halfway in between the 2-week span of time in which the Jactus had grown 3 times as tall. Uditi's ability to imagine some values of the Jactus' height *within* a unit of time became important in later supporting a shift to chunky continuous variation, in which she could think of variation in time as increasing by intervals of a fixed size with an understanding that time values in between were going to be associated with corresponding height values. The activities of iterating and partitioning these complex units may have solidified the new span of time as a unit which can be operated on; thus by the time students had to convert between multiplicative growth for days and growth for weeks, they were able to do so without difficulty.

3.3.3. Developing a scaling image of multiplicative growth

At this stage, all three students could write expressions and equations with exponential notation, such as $2^7 = 128$. However, as seen in Uditi's explanation, an expression such as 2^7 was a way of representing the action of repeated multiplication as iteration. In that sense, even though the students could theoretically write and calculate an expression with a larger exponent, such as 2^{25}, they refrained from doing so. This may have been because none of the students could mentally keep track of repeated multiplication for a large exponent, nor had they developed the idea that they could, in principle, use their calculator to calculate 2^n for any value of n.

Furthermore, the students' reliance on images of repeated multiplication constrained their ability to make sense of non-whole number gaps in time as a measure of growth. We saw this in Uditi's work in Fig. 7, but it also occurred in cases in which students had learned to use exponential notation and write equations such as $b^x = R$. For instance, consider the following task in which students had to determine the plant's height at Week 0.25 (Fig. 9). By the time the students encountered this table, they were adept at determining missing values by taking the ratio of two height values, R, and using that ratio to determine the missing height. For instance, this strategy applied to the table in Fig. 9 would involve taking the ratio of any two height values, because every increase from one week value to the next was the same: 0.25 weeks. The 0.25-week growth factor for the plant's height from Week 1 to Week 1.25 is approximately 1.1892. Therefore one could take the height of the plant at Week 0, which is 1 in., and multiply it by 1.1892 (the 0.25-week growth factor) to find the plant's height after 0.25 weeks.

However, neither Uditi nor Jill used that familiar strategy. Instead, both relied on a more cumbersome approach, writing an equation and guessing and checking. For instance, Jill wrote $y = 1 \times \_\_^{0.5}$ next to the table entry (0.5, 1.412) and then used her calculator to guess and check different numbers she could put in the blank in order to satisfy the equation. In this manner, she determined that the growth factor was 2 and wrote the equation "Height = $1 \cdot 2^{\text{week}}$". Jill could then replace the week value with 0.25, calculating $1 \cdot 2^{0.25}$ to determine the missing height value at 0.25 Weeks. Although the students could

Week	Height
0	1″
0.25	???
0.5	1.41421356237310″
0.75	1.68179283050743″
1	2″
1.25	2.37841423000544
1.5	2.82842712474619

Fig. 9. Table of values in increments of 0.25 weeks.

write expressions such as $1 \cdot 2^{0.25}$ to determine a height value at 0.25 weeks, when directly provided with a growth factor, such as 2, and asked how much a plant would grow in 0.25 weeks, neither student could answer that question by calculating the expression $2^{0.25}$. Why were they able to calculate with an expression such as b^x for non-integer x-values in some cases and not others?

The difference appears to be linked in part to the students' reliance on repeated multiplication images of growth. Uditi and Jill could make sense of the equation "Height = $1 \cdot 2^{0.25}$" to calculate a *static height value*, such as the plant's height at one particular time, 0.25 weeks. They thought of the exponent, 0.25, as representing one particular point in time, which we would conceive of as 0.25 weeks since Week 0, the time when the plant began to grow. They did not conceive of the exponent 0.25 as the representation of a span of time, or 0.25 weeks in elapsed time (a growth in time of 0.25 weeks from any arbitrary point in time). Neither student could therefore make sense of the expression $2^{0.25}$ as a measure of growth, a representation of how many times taller the plant would grow in a span of 0.25 weeks. Thus, when provided a weekly growth factor of 2 and asked to determine how much the Jactus would grow in 1 day, the students could not calculate $2^{(1/7)}$. If one's mental image of exponentiation is repeated multiplication, it is difficult to conceive of multiplying 1/7 factors of 2 and thus make sense of a non-integer exponent as a span of time. Instead, Uditi had to write the equation "$\_\_^7 = 2$", and then guess and check values to determine the number that would satisfy the equation. By changing the time unit from weeks to days, Uditi circumvented the difficulty of expressing growth for a non-integer time span.

In order to encourage alternate images of the growth in the Jactus' height, we returned to the Geogebra script to model the continuous scaling motion the plant undergoes as it grows over time, and asked the students to describe the plant's growth verbally and with gestures. The students provided descriptions such as, "It starts small and gets very large," with an accompanying sweep of their hands to indicate an exponential curve. Students also graphed different exponential functions, and were asked to draw a picture of a general exponential function (rather than a specific function for which they could plot points). Uditi's graph (Fig. 10) differed from Jill's in that she could quickly produce it without making calculations or plotting points; her graph has no specific time marks on the height column. Although the graph alone is not evidence that she had begun to imagine the plant's height varying continuously as time varied, the difference between her graph and the other students' graphs, combined with the fact that prior to this day Uditi's graphs did require plotted points, suggests that she may have begun to imagine growth that relied on an image other than repeated multiplication (Leinhardt, Zaslavsky, & Stein, 1990).

In order to further encourage conceptions of multiplicative growth that were not reliant on repeated multiplication images, we provided students with tasks that had only two data points with a large-week gap in between. For instance, the students had to determine the growth factor of a plant that was 256 in. high at 4 weeks and 1,073,741,824 in. high at 15 weeks. In prior days, the students struggled to make sense of this problem because the 11-week gap was too large to mentally imagine repeated multiplication. Now, however, both Uditi and Jill could take the ratio of the two height values (which is 4,194,304) and then write the equation "$\_\_^{11} = 4{,}194{,}304$" as a way to express the relationship between an elapsed time of 11 weeks and the quotient of the two height values. The students completed multiple tasks of this nature, and because the gaps between weeks were large, they no longer carried out the work of repeatedly multiplying the plant's height. Instead, they truncated the process of repeated multiplication to guessing and checking to find the value of b given that $b^{(x_2 - x_1)} = f(x_2)/f(x_1)$. When asked to describe their images for this equation, the students struggled to verbalize their thinking, but Uditi's explanations in particular no longer referenced repeated multiplication. Instead, she made statements such as, "It's growing in an exponential growth," with an accompanying hand gesture indicating a smooth process of increasing height over time. Although it is difficult to parse her exact meaning, the difference in this description from prior descriptions that referenced repeated multiplication is noticeable. We infer that Uditi may have begun to imagine a scaling or stretching process as the plant grew taller over time. This is a continuous image of growth in height as varying with an increase in time, but it may not necessarily represent an image of continuous growth in height that is exponential; in other words, it was unclear at this point whether Uditi understood that the plant always stretched in height as a multiplicative factor of what came just before.

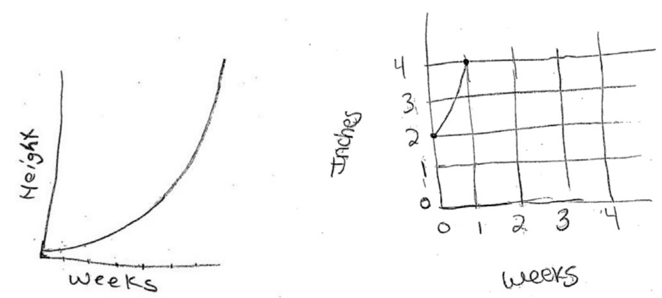


Fig. 10. Uditi and Jill's graphs of a non-specified exponential function.

3.3.4. Student activity fostering scaling images of multiplicative growth

Returning to visual models of growing plants via Geogebra was merely a backdrop to encourage students to create descriptions of growth that did not rely on discrete repeated multiplication actions; the teacher–researcher explicitly encouraged scaling and stretching ideas as an alternate image. It appeared to be easier for students to shift to alternate images if the growth factor of a given plant was not specified; when students knew the growth factor was 3, for instance, it was more natural for them to describe growth as tripling each week. With unknown growth factors in place, students could describe plants as growing in such a way that depicted a sense of continuous scaling or stretching. The process of abstracting images of growth from one particular plant or growth factor may have fostered reflection on the similarity between Geogebra models of growth across multiple scenarios. As described above, Uditi appeared to achieve these alternate images more than Jill or Laura based on her language, gestures, and graphs. Uditi may have been able to imagine continuous growth of the plant's height in the present tense, as indicated by her language rooted in present-tense descriptions, whereas Jill and Laura appeared to still think of growth as repeated copies of height. The introduction of problems with time spans too large to calculate may also have supported a shift to an exponential model that was less reliant on repeated multiplication images. For a large gap in time, such as 25 weeks, students could no longer rely on calculation actions, but instead had to anticipate what that calculation would entail and express it as an exponent, 25.

3.3.5. Coordinating the quotient of $f(x_1 + \Delta x)/f(x_1)$ with corresponding intervals of time $(x_2 - x_1) < 1$

In fostering coordination between the multiplicative growth of y-values with the additive growth of x-values for cases in which time span was less than 1 week, we hoped that students would be able to imagine this coordination for arbitrarily small increments in x; this is Idea 5, the final idea in Section 2.3 guiding our initial thinking about the teaching experiment. Imagining growth for arbitrarily small increments – thinking in any size chunk – would provide a way to understand the meaning of b^x as a measure of growth even when x is not a whole number. One way to introduce this idea was to ask the students to think about how tall the plant would be in the middle of a week; for instance, if a plant doubles between Week 1 and Week 2, how tall will it be at Week 1.5?

One such problem introduced only two data points. The plant in question had a height of 12.513 in. at 2.3 weeks and a height of 13.967 in. at 2.4 weeks. The students were asked to determine how the plant grew every 0.1 weeks and every week. All three students took the ratio of the two height values, which was approximately 1.116, and stated that the plant would grow 1.116 times as tall each tenth of a week. Jill and Laura struggled to use this information to determine how much the plant would grow in 1 week. Uditi wrote "$\_\_^{0.1} = 1.12$", indicating that the blank represented the unknown weekly growth factor of the plant. In contrast to her work on the task in Fig. 9, Uditi could now write an expression with a decimal exponent that represents a measure of growth, not just a static value. By guessing and checking she found that the growth factor was 3, concluding, "It grows by times 3 every week." Beginning to shift to a scaling image of growth in the plant's height could have played a role in supporting Uditi's ability to conceive of b^x as a measure of growth for non-integer values of x.

Additional evidence of Uditi's emerging ability to imagine growth for small chunks of time can be seen in her response to a nearly identical task on a different day, in which the students had to determine, based on a table of values, the growth factor of the plant's height. Working with the first two points, (0.1, 1.116...) and (0.2, 1.246...), all three students determined the ratio of height values to be, once again, approximately 1.116 for an increase of 0.1 weeks. Jill and Laura then reverted to additive reasoning, multiplying the growth factor by 10 to determine how much the plant would grow in 1 week. Uditi, however, was able to correctly relate the growth for 0.1 week to the growth for 1 week, raising the growth factor 1.116 to the power of 10 to find that the plant grew three times as large each week. The teacher–researcher asked Uditi, "How come it's to the tenth power instead of times ten?" and Uditi replied, "Because it grows, like, up for this one [indicates movement from 0 weeks to 0.1 weeks], and then you'll have to do that again to get this one, it's, like, two times to get this one, [indicates from movement from 0.1 weeks to 0.2 weeks] so, and then it goes by ten." This suggests that Uditi may have indeed constructed an image of the plant's height stretching in a scaling-up process such that at each time point (i.e., at 0.1 weeks, at 0.2 weeks, etc.), the height was 1.116 times as tall as it was at the prior time point.

We can contrast Uditi's actions with this problem to the earlier task in Fig. 7 in which she unsuccessfully attempted to obtain a growth factor of 1.55 in four 0.1-week chunks. Although Uditi struggled with her language, it appeared that she was indicating a process of multiplying the height by the growth factor, 1.116, again and again for each 0.1-week increase in time. She understood that because there are ten 0.1-week increments in 1 week, this could be represented as $(1.116)^{10}$. Uditi also understands that 3^x represents how many times taller the plant grows in x amount of time, in this case, 1 week. For 1/10th of x amount of time, or 1/10th of a week, the plant would grow $3^{(0.1)}$, or approximately 1.116, times as tall. In addition, because there are 10 chunks of 0.1-week spans within a week, Uditi recognizes that the plant will grow 1.116^{10} times as tall in 1 week. This represents an understanding of the reciprocal relationship between roots and powers. Uditi may have been able to imagine the plant growing some constant multiplicative amount for each 0.1-week chunk in order to get to 1 week's worth of growth.

On the last day of the teaching experiment, the students encountered a task designed to determine whether they understood that the ratio of $f(x_2)$ to $f(x_1)$ will always be the same for any same $\Delta x = x_2 - x_1$, even when Δx is a fraction. The students studied the table shown in Fig. 8 and had to determine how many times taller the plant would grow in 1 day. Jill and Laura both left the question blank, but Uditi wrote "$3^{.14} = 1.17$", explaining, "It's 1 divided by 7 because it only shows the result for 1 week on the table, and there are 7 days in a week. So I divided 1 week into 7 parts, which represents 1 day each and it's .14 of a week." The teacher–researcher interjected, "Into 7 parts?" and Uditi responded, "And each one is, like, 1 day." Again this reflects an understanding of the reciprocal relationship between roots and powers; Uditi now understands that if the growth factor for an amount of elapsed time x is 3, then the plant will grow $3^{(1/7)}$ times as tall in 1/7th the amount of x. Uditi's description of each part as 1 day could have reflected an image of growth for one day, a smaller chunk of time than a unit, but her language was not clear. Uditi's response, however, stands in contrast to her response to a similar item discussed in Section 3.3.3, when she could not calculate $2^{(1/7)}$ to determine how much the Jactus would grow in 1 day and instead had to write "$\_\_^7 = 2$". At that time, Uditi's reliance on images of repeated multiplication constrained her ability to think of an expression b^x as a representation of growth in x days. By this stage, Uditi could now directly calculate $3^{0.14}$, understanding 0.14 as a span of time, 1/7th of a week (or 1 day), and $3^{0.14}$ as a representation of how much the plant would grow in that amount of time.

3.3.6. *Student activity fostering within-units coordination*

What enabled the shift in Uditi's thinking that allowed her to conceive of exponents as measures of growth rather than static values? Firstly, as discussed in Section 3.3.4, it was important to develop scaling images of growth in the plant's height as varying with time. In addition, we hypothesize that the introduction of a new type of task introduced an element of abstraction that required the students to shift from calculation to anticipation. Typically, the students encountered tasks providing data points in terms of specific time and height values. Students could take the ratio of two height values and compare it to the difference in the corresponding weeks to determine the unknown growth factor. Uditi knew that $b^{\Delta x} = y_2/y_1$, and a repeated multiplication understanding of $b^{\Delta x}$ enabled a mental model of multiplying the plant's height by b repeatedly for the number of weeks represented by the gap from x_1 to x_2.

Tasks introduced toward the end of the teaching experiment were of a different nature. Rather than presenting students with known height and time values and prompting the determination of the growth factor, these tasks instead *provided* the growth factor or a few known values from which students could easily determine it (e.g., Fig. 8). Students could no longer take ratios of height values to answer questions about growth, but instead were asked to predict what those ratios should be. For these tasks, students had to instead anticipate what would happen for different Δx values. The task in Fig. 8 presented a plant that triples each week and introduced questions with height values that were too large to calculate directly by taking ratios. Uditi could not rely on a table, which forced her to instead think only with the known Δx value, the known b value, and to anticipate how she could determine, rather than calculate, the ratio y_2/y_1. The fact that the numbers were too large to directly calculate a ratio may have encouraged a shift to thinking about the relationship between Δx and b.

In contrast, Jill may not have understood that the expression $b^{\Delta x}$ reflects growth in height in Δx amount of time. For instance, given a plant that doubled every day, the only way that Jill could determine its growth in a week was to calculate the plant's height for 14 days and then take the height value at Day 14 and divide it by the height value at Day 7. Jill's work on a similar task reveals her difficulty shifting between units of time. Given a plant that grew 8 times as large in 3 weeks, Jill predicted that it would grow approximately 2.67 times, or $8 \div 3$ times, in 1 week. In this task, Jill actually produced a table of data with the points (0, 1); (1, 2); (2, 4); and (3, 8). Although her own table showed that the plant doubled each week, Jill may not have been able to use that information if she only viewed each table entry as representing a static height value. The point (1, 2) appeared to represent, to Jill, the plant as 2 in. tall at 1 week, rather than also representing a plant that was now twice as tall as it had been the prior week. Jill's focus on the tables were as expressions of correspondence between x and y, rather than as an expression of co-varying growth between height and time.

4. Discussion

The students began the teaching experiment understanding exponential growth as repeated multiplication, but they did not explicitly coordinate repeated multiplication in y with growth in x. In time the students coordinated multiplicative growth in y with additive growth in x, first implicitly without quantifying x, and then explicitly quantifying x as a number of weeks. Negotiating non-uniform tables of values, students then began to coordinate the ratio of y_2 to y_1 with additive increases in x for small week time spans, then engaged in processes of re-unitizing to make sense of larger time spans. The students learned to coordinate the ratio of y-values with $b^{\Delta x}$ for the growth factor b, coming to understand that this relationship would hold for any same Δx. Jill and Laura understood this for Δx values greater than 1, but Uditi was the only student who began to display evidence of within-units coordination, making use of non-natural exponents to represent growth in y for non-whole number increases of x.

Reflecting on the entire set of tasks over the course of the teaching experiment, we identified a number of elements that supported students' shifts toward increased degrees of coordination between multiplicative changes in y-values with additive changes in x-values. Although it was an explicit conceptual goal that students would understand that the constant ratio change in y-values depends on b and on Δx according to the equation $y_2/y_1 = b^{x_2-x_1}$, we did not enter the teaching experiment with an understanding of how to design a full sequence of tasks and activities to foster this understanding, particularly for values in which $\Delta x < 1$. The process of retrospective analysis identified the three major shifts and sub-shifts through which students progressed in order to reach this understanding, which then enabled us to return to the teaching experiment data in order to consider how particular task sequences and teaching actions fostered specific mathematical activity on the part of the students.

In particular, the tasks successively increased the cognitive demands on the students by removing their ability to directly calculate; instead students had to anticipate what the calculations would be and mentally coordinate how the x- and y-values were growing together. This occurred in the drawing tasks and the table tasks when the students encountered missing weeks, requiring them to imagine repeated multiplication actions across two or three weeks without directly carrying out the calculations. It also occurred in the two-data-point tasks that introduced first small gaps and then large gaps between weeks. As the size of the gaps grew, it became too cumbersome to create tables with the missing values, so the students had to anticipate exponentiation without carrying it out and then find a way to coordinate that action with the number of weeks missing between the two points. The final set of tasks also removed Uditi's ability to directly calculate ratios of given height values, instead providing growth factors and asking her to anticipate what the ratio would be for various gaps in weeks. By successively removing the students' abilities to calculate, the tasks encouraged anticipated calculations, which required mentally coordinating the growth in height with the growth in time.

In addition, the task sequences encouraged students to reflect on the common elements of their actions across multiple drawings, tables, or problems. Students encountered, for instance, tables representing the same growth factor but with different time and height values, and noticed what was similar about their repeated multiplication actions for different changes in weeks. The students also encountered different types of tables or two data point problems that all involved the same goal: determine the missing growth factor and the missing initial height. By repeatedly coordinating the growth factor with the additive change in time, the students were able to abstract the relationship $b^{(x_2-x_1)}=f(x_2)/f(x_1)$.

Finally, many of the tasks culminated in implicit and explicit prompts to imagine a general process. The drawing tasks included prompts to describe how tall the plant would be at a given week given its height for the prior week. The table tasks introduced far prediction problems, which encouraged abstraction of $b^{(x_2-x_1)}=f(x_2)/f(x_1)$, and the two data point tasks included prompts to describe how to determine the growth factor given any two time and height values. Because the students had already experienced multiple opportunities to shift from calculation to anticipation and to reflect on the commonality of their actions across multiple problems, they were able to begin to articulate general relationships.

4.1. The covariation approach to exponentiation and continuous variation

Confrey and Smith (1994) described a rate-of-change approach to exponential functions and reported on instances in which students made multiplicative comparisons, interpreting a table of data with a growth factor of b as describing a function that increased by a "constant rate" of b. Our students similarly calculated ratios between successive y-values for constant changes in x-values. In doing so we believe our students engaged in covariational reasoning, as described by Confrey and Smith (Confrey & Smith, 1994; Smith & Confrey, 1994; Smith, 2003), by coordinating sequences of values that represented the plant's height and the amount of time the plant had been growing. This manner of operating, however, can occur within a static image of a quantitative situation. In order for students to begin to think about continuous variation, they must be able to envision two quantities' values varying in tandem, imagining how one variable changes while imagining changes in the other (Saldanha and Thompson, 1998). Thinking about the simultaneous covariation of two quantities must occur for students to engage in smooth continuous variation (Thompson & Carlson, in press).

Smooth continuous variation entails imagining the variation of a quantity's value as its magnitude increases in bits while anticipating that within each bit the value varies smoothly (Thompson & Carlson, in press). We consider this way of thinking to be a challenging goal for middle school participants – in particular for exponential growth – due to how growth is defined geometrically. How the students thought about exponential functions covariationally involved thinking about a next value as the product of a prior value and the growth factor; that way of thinking is inherently chunky. We propose that Uditi's thinking about exponential growth at the end of the teaching experiment was therefore what Thompson and Carlson (in press) describe as chunky continuous variation, thinking of the variation of a quantity's value as increasing by intervals of a fixed size. Uditi could think about time varying for any interval, and she knew that the plant's height varied continuously throughout any interval, and moreover that she could, if necessary, determine the plant's height for any value in between two time values. Further, a shift from the inherently discrete model of repeated multiplication to scaling images for the increase in a plant's height, in which the height increases as a multiplicative factor of what came just before, supported an understanding of the reciprocity between roots and powers. Uditi appeared to imagine the plant growing some constant multiplicative amount for any given chunk of time; for instance, she knew that if the plant grows for a tenth of a week, there is a constant growth factor for a 1/10th time period, and that growth factor raised to the 10th power will give the growth factor for a week. However, there is insufficient evidence that Uditi could imagine the plant's height varying *exponentially* as elapsed time varied simultaneously over any interval.

4.2. The correspondence view of exponential function

A flexible and powerful understanding of exponential function includes a correspondence view, understanding that $y=a\cdot b^x$ represents a relationship in which each value of x is associated with a unique value of y. All three students could write equations expressing the relationship between the plant's height and the number of weeks it had been growing, for instance, writing an equation such as $h = 5 \cdot 2^w$ and describing the growth factor 2 as how much the plant "multiplies each week" and the initial value 5 as "where the growing of the plant starts from at week 0, which has a height of 5." However, we posit that although the students could readily create equations in the form $y = ab^x$, their focus on coordinating multiplicative growth in y with additive growth in x resulted in the equations initially representing a covariation action for the students. For instance, when attempting to find missing growth factors as seen in Section 4, the students coordinated the growth factor with the additive change in time. Consider the difference between $y = b^x$ and $y_2/y_1 = b^{\Delta x}$. The first yields a specific height, y, at a specific time, x. The second represents a ratio of height values given an additive change in time. Although the students wrote equations in the form $y = b^x$, they appeared to think about that equation in terms of the second meaning rather than the first. This changed only when students had to grapple with non-integer values for x. Then, for equations such as "Height $= 1 \cdot 2^{0.25}$", as discussed in Section 3.3.3, students did conceive of the exponent as representing a specific time, rather than a change in time, and the y-value as representing a specific height rather than a change in height. This understanding enabled students to make use of correspondence rules to identify unknown growth factors and unknown height values. Ultimately, however, it was the ability to flexibly shift between those two conceptions of the equation $y = ab^x$ that enabled Uditi to represent and conceptualize growth for non-integer time spans.

4.3. *Proof of concept for middle-school students*

One reason we focused on the middle school population was because students in our participating schools encounter exponential growth in their mathematics classrooms in eighth grade. We were interested in exploring students' evolving conceptual development as they encountered exponential situations for the first time, which necessitated a younger participant group. However, this also resulted in some challenges and constraints in the types of problems and tasks we were able to design. Enabling the students to physically manipulate and visually observe the growing plant with Geogebra limited the growth factors available; a growth factor larger than 4 resulted in numbers too large for Geogebra and the students' scientific calculators, so they only had opportunities to explore plants that doubled, tripled, or quadrupled.

In addition, we typically constrained the growth factor to whole numbers because our participants did not have access to sophisticated algebraic manipulation abilities or the notion of logarithms. This meant that they were limited to guess and check methods for determining the growth factor for mystery plants. For instance, the students often determined the growth factor by taking the ratio of two height values a certain number of weeks apart. Imagine a situation in which the growth factor is approximately 97.66 for a time period of 5 weeks. A high school student could write the equation $b^5 = 97.66$ and then solve for b, calculating $97.66^{1/5}$ in order to determine that the mystery growth factor is 2.5. Because the middle school students did not possess this degree of facility with algebraic manipulation, they instead had to guess and check to determine a number they could multiply by itself 4 times that would equal 97.66. If the growth factor was something other than a whole number, the guess and check method was typically too time consuming to allow for any significant progress within the constraints of the teaching experiment. However, in a future iteration of this work we would introduce simple non-natural growth factors early on, such as 1.5 or 1.25; we suspect that this introduction would foster images of exponentiation that are not reliant on repeated multiplication as iteration.

Despite these limitations, the results presented in this paper offer a proof of concept that even with their relative lack of algebraic sophistication, middle school students can engage in an impressive degree of coordination of quantities when exploring exponential growth. In addition, we have presented evidence that students can generalize their understanding of exponential growth to view b^a as a factors of b for non-natural values of a, as suggested in theory by Weber (2002). We contend that shifting to a scaling image was a critical element in constructing this particular understanding of exponentiation. The Jactus context offered a scenario in which students could begin to make meaningful sense of non-natural exponents as they imagined the height of the plant growing over time.

Contexts such as the Jactus can support students' abilities to imagine two co-varying quantities, directly manipulate the quantities in question, and engage in the act of quantification as they attempt to make sense of how the plant grows over time. The disadvantage of the Jactus scenario is that it is unrealistic, and students must suspend their disbelief in order to enter an imaginary world and engage with the problems in it. This trade-off may be worth it, particularly for younger students, in light of its ability to afford (although certainly not guarantee) students the opportunity to visualize quantities varying continuously. Other more realistic situations, such as exponential growth problems in finance, may not be as tractable for the middle school population. Moreover, more typical middle-school approaches to exponential growth, such as the chessboard problem, may prove too limiting in terms of their ability to foster continuous images of growth.

The common elements we identified in the task sequences also suggest possibilities for future curriculum development for exponential growth. Tasks that progressively shift away from direct calculation and instead require students to engage in increasing degrees of mental coordination and anticipation of calculations can foster a covariational understanding of exponential growth. Repeated exposure to the same mathematical activity also emerged as a key feature; students need multiple opportunities to engage in repeated reasoning (Harel, 2007) in order to have the opportunity to reflect on the commonality in their activity. Furthermore, prompts to describe a general process can further solidify the relationships at hand.

Weber's (2002) theoretical analysis of exponentiation suggested that students could transition to a process understanding by interiorizing their repeated multiplication actions. The results reported in this study support Weber's suggestions, showing it is possible to transition from a naïve understanding of exponential growth as repeated multiplication to ultimately coordinating constant ratios for *y* with additive differences for *x*. Encouraging within-units coordination and linking that perspective to a correspondence view could then ultimately support middle-school students' understanding of exponential growth that would position them well for a rich and connected understanding of functions in high school and more advanced mathematics.

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