6 Principles for Quantitative Reasoning and Modeling

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Modeling the motion of a speeding car or the growth of a Jactus plant, teachers can use six practical tips to help students develop quantitative reasoning.

Encouraging students to reason with quantitative relationships can help them develop, understand, and explore mathematical models of real-world phenomena. But what is quantitative reasoning?

Consider an example. Study the picture shown in the **opening photograph**. What do you see when you look at the picture?

Imagine that a teacher gave that picture to his or her students and asked them to describe and write down everything they saw in the picture. The following three students examine the figure and focus on different aspects:

- Amelia notes that there are two cars, a marker that says "slow," and some buildings in the background.
- Ben notices that the blue car is moving and that the silver car is parked. He also notes that there is a "space" between the two cars, but he does not think any further about how big the space is or how it could be measured.
- Caroline focuses on the length of the blue car compared with that of the silver car, wondering which is greater in terms of feet. She notices that the blur in the picture is caused by the car moving some distance in the amount of time that the camera's shutter was open.

Amelia's conception of the scene is what we would call *non-quantitative*; she did not focus on any aspects of the figure that could be quantified. Ben's conception could be called *proto-quantitative*; how he thought about the space between the two cars is necessary for building a quantitative conception of the situation without actually conceiving of any quantities.

Caroline, in contrast, noticed attributes of the situation, such as the length of each car, and imag-

PHOTO FRAMES: KARANDAEV/THINKSTOCK;
CAR: TONY BUCKINGHAM/LONDON EVENING STANDARD

ined them as measurable. Caroline was thinking quantitatively; for instance, one thing she focused on was how far the car might have moved during the time when the camera's shutter was open. If Caroline were to develop a general way to express average speed, she would need to conceive of a change in one quantity (the distance that the car traveled) in relation to another quantity (the number of seconds that the shutter was open).

QUANTITIES AND QUANTITATIVE REASONING

A *quantity* is a measurable attribute of an object or phenomenon. *Quantitative reasoning* is a way to describe the mental actions of a student who conceives of a mathematical situation, constructs quantities in that situation, and then relates, manipulates, and uses those quantities to make a problem situation coherent. Thompson (2011) argued that quantities are not "out there" in a problem situation; instead, quantities exist in students' conceptions of situations, and their measurement requires quantification.

Quantification (Thompson 2011) is the process of conceptualizing an object (such as a car) and an attribute of the object (such as how far the car traveled) so that the attribute has a unit of measure (such as feet). This process of quantification is essential to constructing a quantity.

Consider a student's understanding of speed. To imagine assigning a speed to an object, he or she must first have conceived of speed as a quantification of completed motion, which entails that there is something about the object's motion that can be measured (Thompson and Thompson 1994). Thompson and Thompson argued that to quantify completed motion, the student must imagine a way to measure it. The measurement of speed relies on conceptualizing two quantities—the distance traveled and the time required to travel that distance—and a multiplicative comparison of those quantities. Then the student must be able to conceive of a value resulting from the measurement process. In this case, speed represents a direct proportional relationship between the distance traveled and the time required to travel that distance. The *value* is the result of the quantification process, such as 55.2 miles per hour. Understanding speed in this way results from using models of a situation (such as a car traveling) and representing its measurable components.

We believe that quantitative reasoning is the foundation that supports students in modeling phenomena mathematically because it provides a means for them to create relationships between quantities that constitute those models. We see quantitative reasoning as a category different from forms of inquiry such as discovery-based learning. It is not a teaching technique per se but a specific way of thinking about mathematics. In other words, a focus on quantities and their relationships means approaching the mathematical topics themselves from a specific mathematical stance, one that emphasizes activities such as measurement, conceptualizing magnitudes, and constructing mathematical relationships.

Because we see quantitative reasoning as a mathematical orientation, we think it is broadly applicable to all teachers interested in supporting modeling in their classrooms, regardless of the type of inquiry they use. Here we describe the relationship between modeling and quantitative reasoning, provide principles for integrating quantitative reasoning in the classroom, and offer examples to demonstrate the utility of these principles for supporting students' mathematical modeling activities.

MODELS AND MODELING

Models are conceptual structures that represent real or imagined scenarios and relate elements of those situations. *Modeling* involves building, testing, and applying conceptual representations of phenomena—a practice that is central to learning and doing science and mathematics. Modeling activities that are grounded in students' conceptualization of quantities are a powerful way to support students' reasoning and sense making as they explore mathematical ideas and relationships.

Both the NextGen Science Standards (2013) and the Common Core State Standards for Mathematics (2010) point to modeling as a central, unifying theme across science and mathematics. Instruction based on modeling begins with the goal to engage students in understanding the world by constructing and using models to describe, explain, and predict the behavior of physical or hypothetical phenomena. We have found the principles shown in **figure 1** to be useful in thinking about how to integrate quantitative reasoning in modeling instruction.

The Six Principles

1. Rewrite a problem situation or prompt so that the students must identify the quantities that they believe are relevant to solving the problem.
2. Ask questions about a problem that focus on why students chose to identify particular quantities and how they intend to or imagine measuring those quantities.
3. Have students identify and test relationships between the quantities that they measure. Push them to justify why those relationships always or do not always hold.
4. Once students have created a model, ask them to determine how varying individual quantities affects the rest of the quantities in the model.
5. Have students develop a representation (physical, visual, etc.) of the situation they are modeling that consists of all the quantities and their interrelationships.
6. Have students revise and retest aspects of their model that may not have been accurate.

Fig. 1 Teachers can use these practical tips to integrate quantitative reasoning into modeling lessons.

IMPLEMENTING THE PRINCIPLES

In this section, we elaborate on the six practical suggestions shown in **figure 1** by describing a scenario from a teaching and research setting. This scenario and others like it demonstrate how a relatively typical problem (the growth of a Jactus plant), and the medium used to present it (a GeoGebra script) can be articulated from a quantitative reasoning perspective and used to support students' modeling activities.

The Problem Context Affords Quantification (Principles 1, 2, 5)

The Jactus plant context (see **fig. 2**) invites students to explore the growth of a plant that increases its height by a constant multiplicative factor (such as doubling) each week. A specially designed GeoGebra script (see **fig. 3**) enables students to explore how the plant grows exponentially over time by changing the parameters (such as the growth factor and the initial height) and then dragging the plant's base (go to **www.nctm.org/mt056**). Although the Jactus scenario is not a realistic problem, it provides an easily visualized context with quantities that students can directly observe and manipulate, in contrast to scenarios such as population growth or compound interest. As students explore the way in which the plant grows, teachers can ask them to

The Jactus Plant Problem

Here is a picture of the Jactus plant. Compare to this picture and draw another picture of what the Jactus will look like after 1 week. Then draw a picture of what the Jactus will look like after 3 weeks.

Fig. 2 A growth problem uses quantities that students can observe and manipulate.

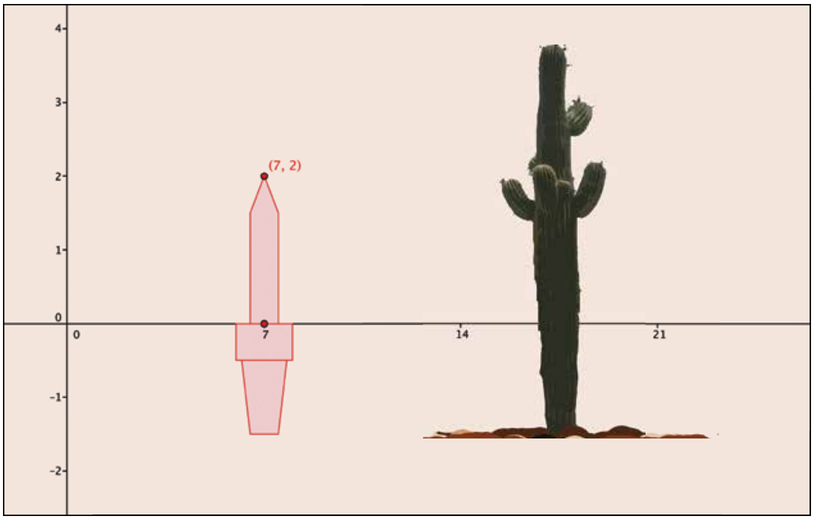

Fig. 3 A GeoGebra script allows for visualization of quantities.

think about the following questions: “What variables do you think might contribute to the Jactus’s growth?” and “How can you keep track of how fast the plant is growing?”

These open-ended prompts encourage students to identify the relevant quantities in determining how the plant grows (see principle 1). Students often mention multiple factors that influence a plant’s growth, such as sunlight, fertilizer, and soil quality, but will begin to isolate the relevant quantities height (in inches) and time (in weeks) when tasked with tracking the plant’s growth. Asking students to think about how they can keep track of growth encourages them to think about how that growth can be measured over time (see principle 2). Asking students to create a picture or diagram of the plant (see principle 5) can also provide a way to assess how they are thinking about the relationship between height and time.

Student Models Help Refine Concepts (Principles 4 and 6)

The modeling task (see **fig. 2**) was introduced to students during a five-week summer school course for incoming ninth-grade students who were preparing to enroll in algebra in the fall. In this course, students met for seventy-five minutes a day and explored three function families—linear, exponential, and quadratic. In small groups of two to four, students worked together on new problems and frequently shared their ideas with one another and with their teacher in a whole-class discussion. All the students in the course had completed general eighth-grade mathematics or prealgebra, but none had experienced exponential functions before the summer school course.

On the second day of the exponential functions unit, before they had encountered equations of the form $y = 2^x$, the students explored a GeoGebra script for a plant with an initial height of 1 inch that doubled each week. They were asked to draw a picture of the plant after one week and after three weeks. **Figure 4** shows Juan’s drawing (**a**) and Paj’s drawing (**b**). The drawings reveal that Paj tried to capture the plant’s height at three weeks as quadruple its height at one week, whereas Juan was not understanding the growth as exponential.

To model the Jactus plant’s growth, we must isolate the two quantities, height and time; determine that height increases multiplicatively as time increases additively; and ultimately coordinate the height ratio of y_2 to y_1 for corresponding x-values representing time for any Δx. For instance, Paj appeared to be able to visualize the height of the Jactus at week 3 as four times as large as the height at week 1, a two-week difference for the function $y = 2^x$. Questions that encourage this coordination are ones that ask students to construct and model relationships between height and time for different Jactus plants (see principle 3). For instance, Paj constructed a table (see **fig. 5**) to represent the growth of the Jactus plant that she drew in **figure 4b**, and Evan represented the relationships in Paj’s table algebraically by writing “height = 2^{week}”.

Once students identify repeated multiplication as the mechanism for determining the plant’s height, they can then begin to think about how varying the initial height and the growth factor will affect the Jactus’s growth (see principle 4). Teachers could ask students to explore a new plant with an initial height of 1/2 inch, rather than 1 inch, and compare the growth of the new and old plants. By creating tables to model the growth of both plants, students can see that the old plant is twice as tall as the new plant at every week value.

This process led one student, Bonito, to revise the previous equation that the students had developed, $y = b^x$, into the form $y = a \cdot b^x$ to account for the initial height (see **fig. 6**). After this discussion, the students always included the initial height, even if it were 1 (principle 6). Encouraging students to justify their representations and to explain what quantity

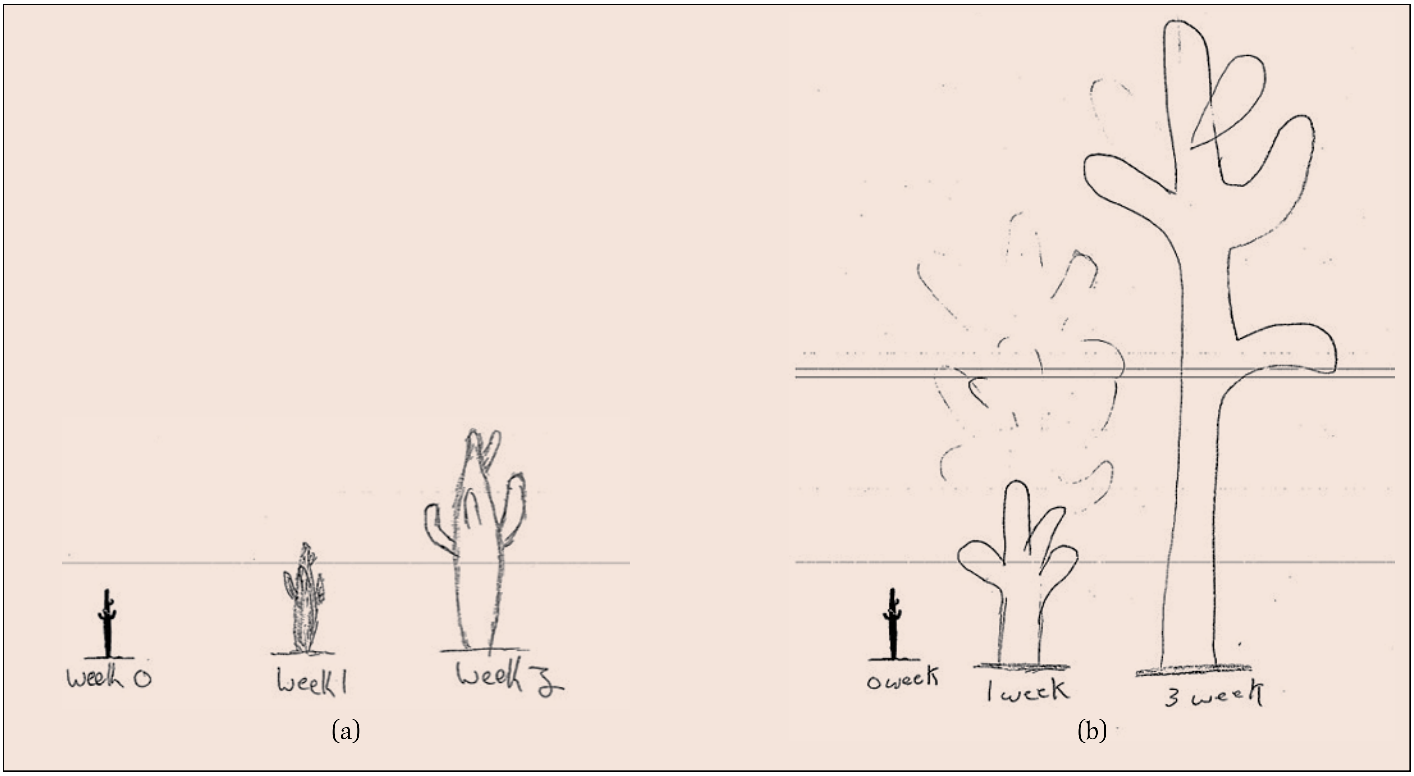

Fig. 4 Students draw the Jactus after one week and after three weeks; the work of Juan (**a**) and Paj (**b**) is shown.

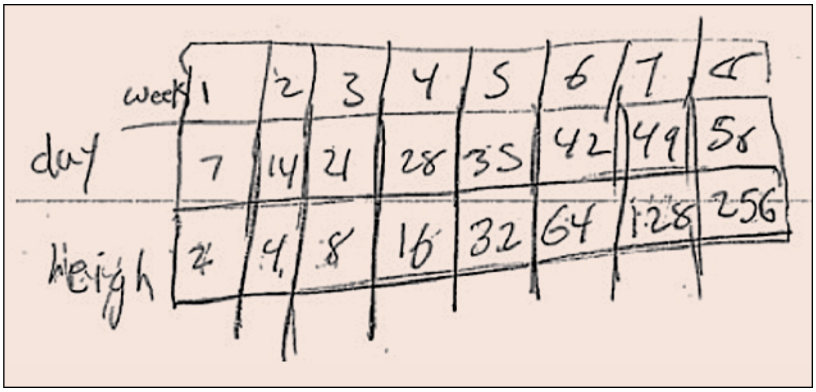

Fig. 5 Paj's table of height and time values represents the Jactus plant's growth.

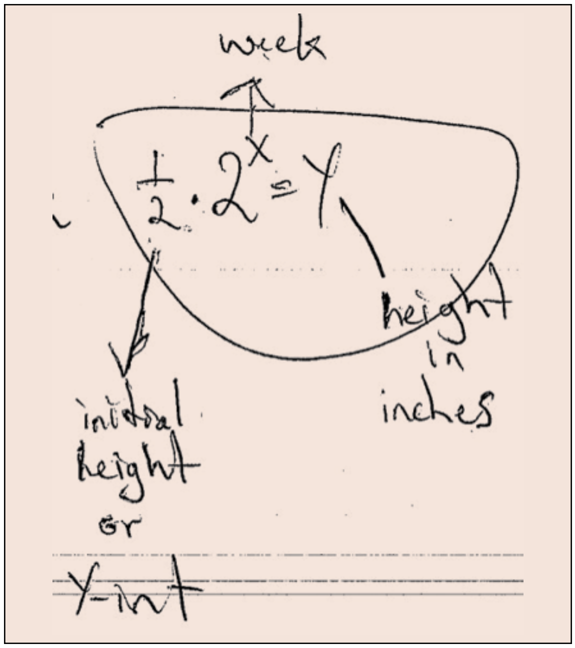

Fig. 6 Bonito's equation for the Jactus plant's growth uses an initial height of 1/2 inch.

or relationship is represented by each variable or parameter in their equations supports a more meaningful link between the situation and the models.

Once students are comfortable with the models that they have constructed, introducing data in different forms encourages students to revisit aspects of their models and rethink how to relate quantities in new ways (see principles 4 and 6). For instance, teachers could introduce tables of data with missing values or gaps between the weeks (see **fig. 7**). In this case, Bonito attempted to coordinate the increase in height with the increase in time (see **fig. 7a**), determining that the plant would grow four times as large in a two-week interval (notice that Bonito wrote "2" between week 3 and week 5 and then wrote "× 4" between 24 inches and 96 inches). However, he decided that the growth factor for three weeks would be 12 rather than 8 (as seen in Bonito's writing "× 12" between 96 inches at 5 weeks and 768 inches at 8 weeks). In contrast, Carter's work (see **fig. 7b**) shows his coordination of the growth in height for three weeks as represented by 2^3, or $2 \cdot 2 \cdot 2$. This correlation is evidenced by the fact that Carter wrote "$2 \cdot 2 \cdot 2$" between the height at week 10 and the height at week 13.

Teachers can encourage discussion and exploration of the relationships between height and time for different "chunks" of weeks to help students see how their exponential models hold, regardless of the value of Δx. They could also examine the students' models to see whether they would still hold for non–whole number exponents—for instance, by asking students to consider the plant's height at day 8 (rather than week 8). Once students can make calculations such as $3 \cdot 2^{(1+1/7)}$ to find the plant's height for a given day, they will begin to see the exponent x in $y = ab^x$ as representing any amount of time, rather than limited to a whole-number week value. The GeoGebra script can also support this conception; students can drag the base of the plant smoothly across the x-axis to observe the plant's height change continuously rather than in discrete chunks. By situating questions and explorations within the world of the Jactus plant and its quantities of height and time, teachers can help students develop meaningful models and representations of abstract concepts such as continuous exponential growth.

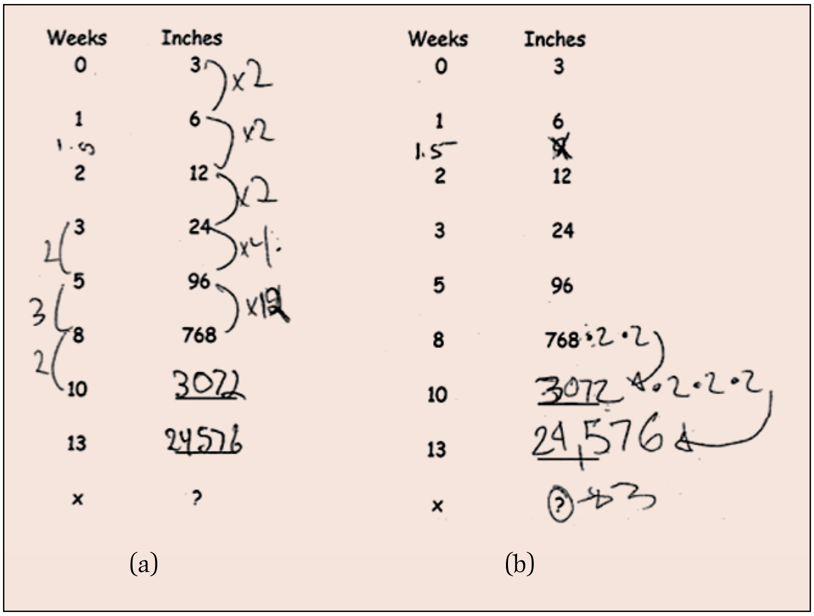

Fig. 7 A table of heights with nonuniform time increments shows Bonito's work (**a**) and Carter's work (**b**).

EXTENDING THE PRINCIPLES

Four Ways to Modify Tasks

The six principles described earlier focus primarily on engaging students in quantitative reasoning through patterns in modeling problems. The importance of modifying textbook exercises and classroom tasks in ways that support these six principles led us to modify our own classroom tasks in the following ways.

1. Ensure that a task describes a situation in which the students themselves must identify quantities relevant to solving the problem rather than prescribing the quantities for them.
2. Create tasks in which the students must attend to the measures of the quantities in the problem as they determine relationships between those quantities. Make attention to the units important by using nonstandard measurements or asking the students to introduce their own measurements.
3. Create natural subparts to a task in which students must articulate their model for a situation and the quantities that constitute it. Doing so allows the students to reflect on the steps they took to solve the problem and to identify natural points at which to rethink their approach.
4. Introduce follow-up questions to tasks that create opportunities for the students to revise their models. These questions often arise from observing students struggle with a particular part of a problem.

Connections to Research

These two examples—the moving car and the growing plant—demonstrate how the principles we have proposed about quantitative reasoning in support of modeling can be enacted in the classroom. Beyond the evidence and experiences presented here, research has also identified quantitative reasoning as foundational to students' ability to think about functions, rate of change, generalization, and proof (Ellis 2009; Oehrtman, Carlson, and Thompson 2008; Weber 2012; Weber et al. 2012). This research suggests that the inclusion of modeling in textbooks and more recently in the Common Core State Standards for Mathematics requires teachers to think carefully about how to prepare and revise lessons to focus on modeling. The six principles we have proposed provide a means for teachers to think about modifying and adapting existing problems to support students' abilities to model through quantitative reasoning.

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