Journal for Research in Mathematics Education
2011, Vol. 42, No. 4, 308–345

Generalizing-Promoting Actions: How Classroom Collaborations Can Support Students' Mathematical Generalizations

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Generalization is a critical component of mathematical activity and has garnered increased attention in school mathematics at all levels. This study documents the multiple interrelated processes that support productive generalizing in classroom settings. By studying the situated actions of 6 middle school students and their teacher–researcher working on a 3-week unit on quadratic growth functions that can be represented by $y = ax^2$, the study identified 7 major categories of generalizing-promoting actions. These actions represent how teachers and students can act in interaction with other agents to foster students' generalizing activities. Two classroom episodes are presented that identify cyclical interaction processes that promoted the development and refinement of generalizations. The results highlight generalization as a dynamic, socially situated process that can evolve through collaborative acts.

Key words: Algebra; Classroom interaction; Collaborative learning; Constant comparison methods; Functions; Intermediate/middle grades; Qualitative methods

Generalizing is widely acknowledged to be a critical component of mathematical activity (e.g., Kaput, 1999; Reid, 2002; Sfard, 1995), with researchers and policymakers arguing that students should learn to generalize in all areas of mathematics (National Council of Teachers of Mathematics, 2000). The role of generalization has enjoyed increased prominence particularly in algebra classrooms, in which calls for reform state the need to develop algebraic reasoning in terms of generalizing activities (Kaput, 1999; Kieran, 2004). Carpenter and Franke (2001) point to the value of establishing generalizations in promoting algebraic understanding: "When students make generalizations about properties of numbers or operations, they make explicit their mathematical thinking. Generalizations provide a class with fundamental mathematical propositions for examination" (p. 156). Publishers have

This research was supported by the National Science Foundation (NSF) under REC 0529502. The opinions expressed in this article are those of the author and do not necessarily reflect the position, policy, or endorsement of the NSF. The author wishes to thank Paul Grinstead and Caroline Williams for their assistance with data collection and for the fruitful discussions that followed.

responded to this emphasis by producing curricula aimed at promoting generalizing activities (e.g., Coxford et al., 1998; Lappan, Fey, Fitzgerald, Friel, & Phillips, 2002), and a number of recent studies have focused on supporting students' generalizing in the classroom (e.g., Jurow, 2004; Lannin, Barker, & Townsend, 2006; Becker & Rivera, 2006).

Despite the importance of generalization to mathematical and algebraic reasoning, its processes are not well understood. Studies have demonstrated students' difficulties in creating correct general statements (English & Warren, 1995; Kieran, 1992; Lannin, 2005), attending to patterns that are generalizable (Blanton & Kaput, 2002; Lee, 1996), and using generalized language (Mason, 1996). Research examining students' generalizing practices in algebra settings has further identified a number of challenges: Students struggle to generalize patterns that are algebraically useful (Lee & Wheeler, 1987; Orton & Orton, 1994; Stacey, 1989), they focus on covariational patterns more readily than the correspondence relationships that allow generalizing the *n*th case (Pegg & Redden, 1990; Schliemann, Carraher, & Brizuela, 2001; Szombathely & Szarvas, 1998), and they experience difficulty shifting from recognizing patterns to generalizing them (English & Warren, 1995; Stacey & MacGregor, 1997). Although students' difficulties with generalizing have been well documented, additional research on how to promote successful generalizing in classroom settings is needed in order to better help teachers support their students' generalizing processes. As Jurow (2004) noted,

> It is often the case that teachers, curriculum designers, and textbook authors fail to recognize that general mathematical patterns are not directly perceptible. Mathematics students do not unproblematically see general patterns through exposure to or experience with multiple, similar cases. Rather, they need to orient to and be guided to recognize what is relevant in and across situations. (p. 297)

Researchers have identified a number of social and pedagogical influences on students' generalizing activities. For instance, Koellner, Pittman, and Frykholm (2008) detailed task and classroom factors that influenced how middle school students engaged with a generalization problem. They found that working with an open-ended problem with many entry points, having opportunities to visualize a concrete representation of the problem situation, and being able to work collaboratively fostered the students' generalization processes. Furthermore, the teacher's discursive actions of pushing for algebraic generalizations without supplying strategies or answers appeared to promote productive generalizing. Research on teachers' pedagogical strategies more generally has identified a number of potentially productive actions for fostering generalization, including highlighting the role of conjecture and justification in classroom discussion, providing access to physical or visual representations of mathematical relationships, revoicing to elaborate or refine student contributions, and encouraging reflection on students' activity (Cobb, Boufi, McClain, & Whitenack, 1997; Hall & Rubin, 1998; Jurow, 2004; Lehrer, Strom, & Confrey, 2002; O'Connor & Michaels, 1996). Fewer studies focus on student interactions, but one such study by Jurow (2004) identified the notions of linking and conjecturing as two participation frameworks that supported students' generalizing in small-group interaction.

Students can be adept at making all kinds of generalizations, but this does not mean that they will be able to generalize in ways that are productive in terms of being mathematically useful, or helpful in achieving the mathematical goals of a lesson or a unit (Lee & Wheeler, 1987; Orton & Orton, 1994; Stacey, 1989). Documenting the specific processes that support productive generalizing is an important step toward understanding how to help students generalize in classroom settings and in understanding the actions in which teachers can engage to foster those generalizations. The study reported in this paper examines how actions and interactions can work in concert with one another to promote generalizing. Moving beyond an exclusive focus on the teacher's pedagogical moves, this study also addresses the types of student actions and interactions that can foster generalizing. This article highlights generalizing as a dynamic process that can evolve through interaction, and it presents two interaction cycles in which particular types of student and teacher acts fostered the development and refinement of new generalizations.

GENERALIZING AS A SITUATED PHENOMENON

Researchers historically have approached the notion of generalizing from a number of perspectives, viewing it as identifying commonalities (Dreyfus, 1991), extending one's range of reasoning beyond the immediate case at hand (Dubinsky, 1991; Harel & Tall, 1991), or shifting from individual cases to the patterns, relationships, and structures across them (Kaput, 1999). Research has also distinguished types of generalizing, either by categorizing types of generalizing activities (Davydov, 1990; Harel & Tall, 1991; Krutetskii, 1976; Piaget & Henriques, 1978; Stacey, 1989) or by developing constructs delineating multiple levels of generalizing strategies (Garcia-Cruz & Martinón, 1997; Lannin, 2005). Taken across the body of research as a whole, these distinctions attend to many dimensions. They account for generalizing as a static model of applying prior knowledge versus generalizing as a dynamic reconstruction of knowledge (Harel & Tall, 1991; Piaget & Henriques, 1978), generalizations that are locally useful versus those that represent more global rules (Stacey, 1989), and generalizing as subsuming particular cases into a general concept versus developing a general concept from particular cases (Davydov, 1990; Krutetskii, 1976). Other constructs elaborating multiple levels of generalization offer descriptions of students' strategies as they engage with particular types of problems (Lannin, 2005) or identify the different cognitive actions in which students engage while generalizing (Ellis, 2007).

These frameworks share a common theme in that generalization is largely viewed as an individual, cognitive construct. This perspective has proved useful in distinguishing types of generalizations, describing levels of generalization as students progress mathematically, and identifying students' competencies and difficulties with generalizing. At the same time, an increasing number of researchers have examined generalizing as a collective act, distributed across multiple agents set in a specific social and mathematical context (Reid, 2002; Tuomi-Gröhn & Engeström, 2003). This perspective attends to how social interactions, tools, personal history, and environments shape people's generalizing actions, conceiving of generalizing as a social practice rooted in activity and discourse (Jurow, 2004; Latour, 1987). It is within this tradition that I define generalizing as an activity in which people in specific sociomathematical contexts engage in at least one of three actions: (a) identifying commonality across cases, (b) extending one's reasoning beyond the range in which it originated, or (c) deriving broader results from particular cases. Throughout this article I will use *generalizing* to refer to any of these three processes or actions, whereas *generalization* will refer to the product or outcome of these actions.

The current reform movement in mathematics education places considerable emphasis on the role that classroom discourse can play in supporting students' conceptual development (Cobb et al., 1997). Accordingly, there have been calls for research that attends to how learners develop understanding by means of participation (Rivera & Becker, 2004). In addressing these calls, this study employs the interactionist perspective (Bauersfeld, 1995; Blumer, 1969), taking learning as a social process that occurs in the interaction among classroom participants. As Bauersfeld (1995) described, "Teachers and students jointly and interactively produce certain regularities and norms of speaking and acting mathematically. Thus, we understand the development of mathematizing in the classroom 'as the interactive constitution of a social practice'" (p. 150). By taking the interactionist perspective, this study focuses on the interactions that occur within the microculture of a classroom, with the negotiations that occur during interactions viewed as mediating between cognition and culture.

This study considers classroom situations through the lens of multiple processes of interactions, in which the students and the teacher co-contribute to the development of meaning through their talk, shared activity, and engagement with artifacts. The interactionist perspective privileges both teacher–student interaction and student–student interaction, which allows researchers to take into account how teachers and students develop shared ways of interacting in ways that support generalizing. The focus of study is therefore on the interactions between individuals within a particular sociomathematical culture (Bruner & Bornstein, 1989). From this perspective, generalization (like abstraction) "is not an objective, universal process but depends strongly on context, on the history of the participants, on their interactions, and on artefacts available to them" (Dreyfus, Hershkowitz, & Schwarz, 2001, p. 378). In contrast to a discourse perspective, the interactionist stance mediates between individualism and collectivism by focusing simultaneously on the dynamics of classroom situations and the mathematical meanings students make (Voigt, 1995). This does not preclude from the study individual construction of meaning; instead, it situates that construction as taking place in interaction with the classroom culture (Cobb & Bauersfeld, 1995).

METHOD

Setting and Participants

The study was situated at a public middle school (students of ages 11–14) located in a midsized city. The participants were 6 eighth-grade students (ages 13–14) who were enrolled in prealgebra (3 students), algebra (2 students), or geometry (1 student). The students' teachers identified them as either high, medium, or low based on their assigned mathematics class, as well as on their mathematics grades, attendance, and participation in class. The group of 6 participants was selected in order to produce a group of mixed backgrounds. It contained 2 students identified as high, 2 identified as medium, and 2 identified as low. There were 3 girls and 3 boys; 1 student was Indian American, 2 students were Asian American, and 3 students were Caucasian. One student was an English language learner. Gender-preserving pseudonyms were used for all participants.

The Teaching Experiment

The students participated in a 15-day teaching experiment (Cobb & Steffe, 1983). The author was the teacher–researcher (identified as "TR" in the transcript excerpts that follow), and two project members familiar with the teaching-experiment methodology and the goals of the project observed each teaching session. Each teaching session lasted 1 hour, and each student also participated in an hour-long preinterview and postinterview. The project members operated two video cameras during the teaching sessions to capture both the whole-group discussions and the small-group interactions for all the teaching-experiment sessions. The project team met every day after the teaching-experiment session in order to debrief and informally discuss what had occurred during the session. An undergraduate student transcribed all the videotaped data using the Transana software program.

One of the goals of the teaching experiment was to examine the factors promoting students' generalizing in the context of exploring situations about a particular type of quadratic growth representing the relationships between the height, length, and area of rectangles that grow proportionally by maintaining the same height-to-length ratio as they grow. The relationship between height, x, and area, y, can be expressed as $y = ax^2$, and the students did not explore more general quadratic relationships of the form $y = ax^2 + bx + c$, with $a \neq 0$. Students worked with computer simulations of growing rectangles in The Geometer's Sketchpad, drew their own rectangles, and justified their generalizations about the nature of the quadratic growth they were examining. The students created tables of values comparing the heights and areas of rectangles and developed multiple generalizations about the constant second differences in their tables. This approach follows the tradition of Davydov (1975) and Thompson (1994) that reasoning with quantities and their relationships should be the basis of algebraic reasoning. However, unlike Davydov's (1975) approach with very young children that emphasizes unspecified quantities, the students in the teaching experiment identified specific numbers and progressed from the specific to the general.

One purpose of the small-scale teaching experiment (cf. Steffe & Tzur, 1994; Thompson, 1994) was to gain direct experience with students' mathematical reasoning through the construction and continual revision of hypothetical learning trajectories (Simon, 1995). The teaching-experiment setting allowed for the creation and testing of hypotheses in real time while engaging in teaching actions, which means that the mathematical topics for the entire set of sessions were not predetermined but instead were created and revised on a daily basis in response to hypothesized models about the students' mathematics. Figure 1 provides a brief overview of the topics addressed in the teaching experiment.

Day	Mathematical topics	Day	Mathematical topics
1	Measurement and area	9	Justifying the second differences as $2a$
2	Comparing perimeter and area	10	Identifying second differences for tables with different Δh values
3	Identifying first and second differences in tables	11	Connecting equations, tables, and graphs
4	Connecting first and second differences to area	12	Graphing parabolas
5	Identifying height:length ratios and creating $y = ax^2$ equations	13	Graphing first and second differences
6	Creating generalizations about second differences	14	Creating $y = ax^2 + c$ equations and graphs
7	Justifying generalizations about second differences	15	Summarizing generalizations
8	Creating $y = ax^2$ equations from tables and identifying the second difference as $2a$		

Figure 1. Overview of the teaching-experiment sessions.

Data Analysis

Data analysis followed the interpretive technique, in which instances of classroom interactions promoting generalizations were induced from the data (Strauss & Corbin, 1990). Analysis relied primarily on full transcripts and secondarily on the video recordings themselves, which allowed the project team to consider mainly the participants' talk, but also their gestures, intonations, and their use of artifacts, drawings, and physical objects. Methods consistent with interactionism involve detailed description and interpretation of transcripts, which are analyzed as documents describing processes (Voigt, 1995). In this case the transcripts were first analyzed to identify instances of generalizing as they fit the previously stated definition: (a) identifying commonality across cases, (b) extending one's reasoning beyond the range in which it originated, or (c) deriving broader results from particular cases. Once all instances of generalizing were identified, the transcripts were then coded via open coding in order to develop preliminary codes describing the talk and actions that preceded and appeared to prompt the generalizations. Actions were coded as fostering generalizing if (a) generalizing occurred in direct response to an action, (b) a generalization mirrored or responded to a new idea introduced by the action, or (c) a conceptual chain could be identified linking the ideas or strategies introduced by an action and a generalization that followed it. The project team[1] met to argue any boundary cases or uncertain instances, and these cases were ultimately decided through a return to the videotaped data to track the proposed chain of actions. Although it is impossible to definitively identify an action as causing a generalization, these criteria provided a basis for making decisions about borderline cases.

In the second round of analysis, referred to as *axial coding* (Strauss, 1987), the codes were related to one another in order to identify a set of causal relationships. Coding was aimed at describing the conceptual relationships between the categories of interactions and the students' generalizing activities, identifying how particular types of interactions may have promoted students' generalizing. For instance, one of the codes described in the section below is *revoicing*, in which a teacher or a student restates another member's generalization in order to elaborate or revise it. Revoicing was considered in relationship to all the generalizing actions that succeeded it in order to hypothesize ways in which the particular act of revoicing appears to foster generalizing. Through considering these relationships, hypotheses emerged (e.g., revoicing actions appear to prompt students to revise or refine their existing generalizations, supporting the development of more explicit or better-defined statements).

As the codes began to stabilize, the data were revisited to begin to approach saturation (Strauss & Corbin, 1990), in the sense that revisited data excerpts reproduced the same categories that had been developed, rather than resulting in the development of new categories or an adjustment of existing categories. The project team met to engage in sample coding of data excerpts until no new properties or relationships emerged from recoding the data. Once the codes were organized into major categories, a final round of analysis occurred in which the transcripts were revisited in order to find all instances of the actions that were coded as fostering generalizing, regardless of whether the action prompted a generalizing activity. To do this, Chi's (1997) method of verbal analysis was used, which is a method for

[1]Aside from the author, the project team's role was limited to assistance in the sense that (a) they videotaped sessions, (b) they engaged in debriefing discussions with the author after each session (which were not used as data), (c) the author would bring uncertain coding cases to the weekly project meeting, and the team would discuss the cases and return to the data to make a joint decision, and (d) they helped with approaching saturation in project meetings, using the coding scheme for new data pieces. The author created and revised the codes but discussed some aspects of them with the project team.

"quantifying the subjective or qualitative coding of the *contents* of verbal utterances" (p. 272, emphasis original). This began with segmenting the data to identify two units of analysis: the generalizing-promoting actions and the generalizations themselves. Each generalizing-promoting action was identified and then tracked in relationship to the number of times it was connected to a generalization or an act of generalizing. Those comparisons are presented in the Results section.

Consistent with the recent approaches of other researchers investigating discourse (Pierson & Whitacre, 2010; Temple & Doerr, 2010), the codes for the generalizing-promoting actions do not distinguish between the teacher–researcher's utterances and the students' utterances. This is consistent with the study's use of the interactionist frame, in which the classroom microculture is brought forth jointly by the teachers and the students (Cobb & Bauersfeld, 1995). The classroom episodes are construed as processes of interaction, in which both the students and the teacher–researcher contribute to a shared understanding. Each of the codes represents an action that occurs within the framework of interacting with other members of the classroom community, regardless of who originated the action.

RESULTS

The results are presented in two sections. The first section introduces categories of generalizing-promoting actions and provides brief excerpts defining and exemplifying each of the actions. The second section identifies and describes in-depth processes of interaction that promoted the development and refinement of generalizations. That section presents two extended excerpts in which the students created and adjusted their generalizations through interaction.

Part I: Categories of Generalizing-Promoting Actions

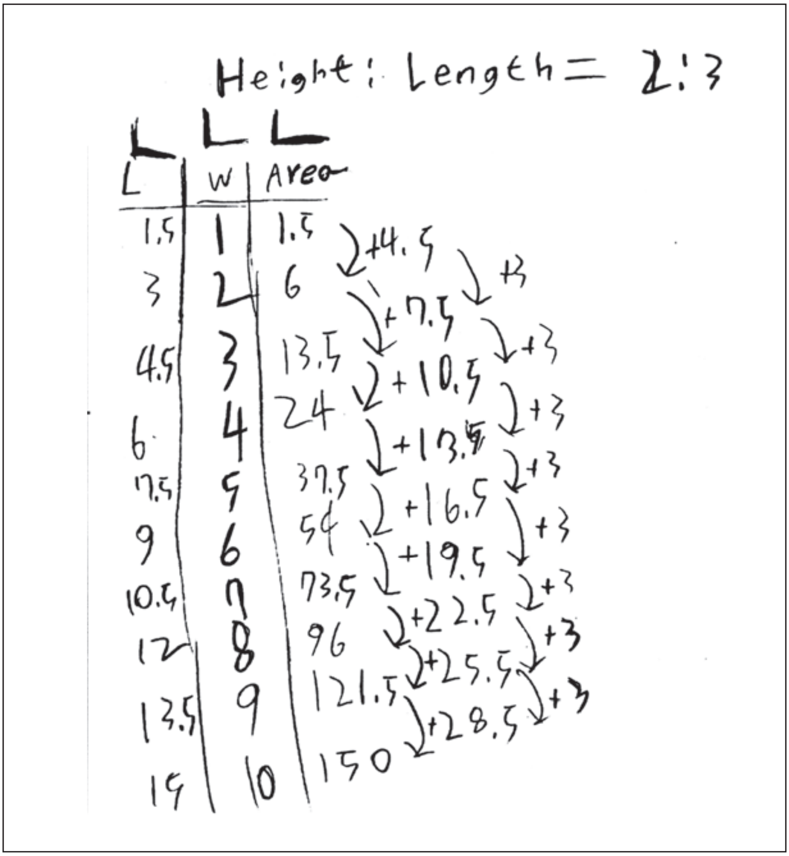
Seven categories of actions emerged from the analysis. Each category describes a type of action or talk that preceded and appeared to foster the development or refinement of a generalization. The 402 action codes are not limited to teacher moves, but instead represent the ways that teachers, students, problems, and artifacts can act in interaction with other agents to promote students' generalizing activities. The leftmost column in Figure 2 describes the major categories of actions, and the two rightmost columns report the number of instances in which each action was connected to a generalization (Yes) and the number of instances in which it was not connected to a generalization (No). The categories are not necessarily mutually exclusive, and some actions could be coded in more than one category. In those cases, the actions were coded in a primary category according to the researcher's perception of the actor's intention and the manner in which the other members of the classroom community responded to the action.

A note about the mathematics. The students participating in the teaching experiment examined the relationships between the lengths, heights, and areas of rectangles

<table>
<tr><th colspan="2">Generalizing-Promoting Action</th><th>Yes</th><th>No</th></tr>
<tr><td colspan="2">Publicly Generalizing:
A member of the classroom community publicly engages in generalizing. This may take the form of—
(a) creating an association between two or more problems, objects, situations, or representations;
(b) identifying an element of similarity across cases; or
(c) expanding a pattern, idea, or relationship to reach beyond the case at hand.</td><td>50</td><td>4</td></tr>
<tr><td rowspan="4">Encouraging Generalizing:
Encouraging others to engage in generalizing</td><td>Encouraging Relating: Prompting the formation of an association between two or more entities</td><td>26</td><td>12</td></tr>
<tr><td>Encouraging Searching: Prompting the search for a pattern or relationship</td><td>14</td><td>2</td></tr>
<tr><td>Encouraging Extending: Prompting the expansion beyond the case at hand</td><td>18</td><td>2</td></tr>
<tr><td>Encouraging Reflection: Prompting the creation of a verbal or algebraic description of a pattern or rule</td><td>25</td><td>3</td></tr>
<tr><td colspan="2">Encouraging Sharing of a Generalization or Idea:
Asking or encouraging another member to publicly share a generalization, representation, solution, or idea. This may occur in the form of formal or informal requests for sharing broadly or restating ideas.</td><td>19</td><td>0</td></tr>
<tr><td colspan="2">Publicly Sharing a Generalization or Idea:
Sharing another member's generalization, idea, strategy, or representation with the larger classroom community. This may take the form of revoicing (restating another member's generalization) or publicly validating or rejecting another member's generalization.</td><td>50</td><td>25</td></tr>
<tr><td colspan="2">Encouraging Justification or Clarification:
Encouraging a member to reflect more deeply on a generalization or an idea by requesting an explanation or a justification. This may include asking members to clarify a generalization, describe its origins, or explain why it makes sense.</td><td>67</td><td>25</td></tr>
<tr><td colspan="2">Building on an Idea or a Generalization:
Building on another member's idea, conclusion, or generalization. Building actions can take the form of refining an idea or using it to create a new idea, rule, or representation.</td><td>14</td><td>1</td></tr>
<tr><td colspan="2">Focusing Attention on Mathematical Relationships:
Directing attention to particular aspects of a problem or representation. A member may direct others' attention to specific mathematical features of a problem or activity.</td><td>31</td><td>14</td></tr>
</table>

Figure 2. Summary of generalizing-promoting actions and connections to generalizations.

that grew while maintaining their length/height ratios. They worked with a script in The Geometer's Sketchpad to explore what happened to the dimensions of a particular rectangle (for instance, a 2 cm by 3 cm rectangle) as it grew and shrank. The script allowed the students to expand the rectangle by dragging a point, and the program allowed the students to measure the height, length, and area of the rectangle at any given size. The students created their own tables of data to represent the phenomena they observed. An example of one student's data table is given in Figure 3.



Height: Length = 2:3

L	W	Area		
1.5	1	1.5	+4.5	
3	2	6	+7.5	+3
4.5	3	13.5	+10.5	+3
6	4	24	+13.5	+3
7.5	5	37.5	+16.5	+3
9	6	54	+19.5	+3
10.5	7	73.5	+22.5	+3
12	8	96	+25.5	+3
13.5	9	121.5	+28.5	+3
15	10	150		

Figure 3. Student's table of data representing a growing rectangle that maintains height/length ratios (*W*/*L*, here).

Because the second differences in the students' well-ordered tables[2] were constant, the students focused their attention on these differences and attempted to make sense of their origins in relationship to the rectangles they studied. They called

[2]By *well-ordered tables*, I refer to tables in which the *x*-values (values of independent variables) increase by a uniform amount, such as 1 cm or 5 cm.

the first differences the *rate of growth* (RoG) of the area of the rectangle when the height increased by a uniform amount. (As shown in Figure 3, the RoG is 4.5 square units, 7.5 square units, 10.5 square units, and so forth for successive 1.5-unit increases in the length.) The students called the second differences the *difference in the rate of growth* (DiRoG) of the area. As shown in Figure 3, the DiRoG is 3 square units for every 1.5-unit increase in the length. Because the students were interested in predicting what the DiRoG would be for different types of tables, they also attended to how the entries in the tables were ordered. In some tables, the height (or length) values grew by 1 cm; in other tables, they grew by different constant height (or length) values, such as 5 cm or 10 cm. The students also encountered tables introduced by the teacher–researcher in which the heights and lengths did not increase by a constant value (such as those in the table in Figure 9 on page 329). In order to focus attention on these increments, the students used the terms *difference in the height*, which they called *DiH*, and *difference in the length*, which they called *DiL*.

Publicly generalizing. Publicly generalizing refers to instances in which members of the classroom community engage in the process of generalizing in a public manner, by sharing ideas and results with others or by generalizing aloud in a collaborative manner. Publicly generalizing can take a number of forms, depending on the type of generalizing in which students engage. For instance, one student, Bianca, remarked on her classmate's equation, $DiL/DiH = a$, after he wrote it on the board. Bianca remarked, "Isn't $DiL/DiH = a$ the same thing as mine, as the *DiRoG* over 2 equals a?"[3] By engaging in the generalizing action of *relating* (Ellis, 2007) her classmate's equation as identical to her own, Bianca was publicly generalizing by creating a relationship of similarity across the two equations. This prompted another classmate, Ally, to publicly generalize by relating the two equations, concluding, "So that means that *DiRoG* over 2 equals *DiL* over *DiH*."

In a different type of public generalizing, Jim hypothesized a general way to find an equation for a function given the DiRoG from a table:

Jim: Well, finding the difference in the rate of growth, you can pretty much go from the difference in the rate of growth backwards to find out what you want to know. Like you can find out any table by going from the difference in the rate of growth backwards. So like, let's say, your difference in rate of growth is 7, you can go back with two numbers like, you know, 7 and 14 or whatever. And you can go back . . . you know, keep going back until you find the original table. For any table you want.

Jim appeared to describe a potential way to determine the values in a height/area table given knowledge of the DiRoG. Jim may have been hypothesizing that because one can calculate the DiRoG directly from the area values in a well-ordered table, it should be possible to go "backwards" to determine area values and even

[3]The two equations are equivalent only when the DiRoG is calculated from a table in which the height values increase by 1; see the Appendix for a more in-depth discussion of the accuracy of these generalizations.

height or length values, given the DiRoG. Bianca objected to Jim's method, and in reaction to it came up with her own way to find an equation:

Bianca: So, say the length is 1 and the width equals 2, and then the length is 2 and the width equals 4 [creating a small length/width table], then this [waving her hand over the length column] is L and this [waving her hand over the width column] is W, so it'd be like L equals 1/2 of W [writes $L = (1/2)W$], or $2L$ equals W. So *this* is just the next step to finding the equation. You can do it with area too.

Bianca did not believe Jim's hypothesis that one could derive the values in any table given the DiRoG. In reacting to Jim's public generalizing, Bianca described a way to find an equation relating the length/width columns of a table, thus creating her own generalization.

Encouraging generalizing. Encouraging generalizing refers to actions in which a member of the community directly encourages a generalizing action on the part of another community member. One can encourage generalizing in a number of different ways; for instance, when the teacher–researcher asked "What's the same about these rectangles?", "What pattern do you notice in this table?", and "Do these generalizations work even if you have decimals?", she prompted the students to engage in different types of generalizing actions by relating, searching, and extending, respectively. Encouraging generalizing was not limited to the teacher–researcher's actions. Students also engaged in encouraging; for instance, when Jim and Tai worked together on a problem, Tai attempted to develop a pattern and Jim prompted him to generalize by asking, "Can you come up with an equation, though?"

One particularly powerful form of encouraging generalizing occurred when a member of the classroom community or a problem context prompted students to predict the outcome of an experiment or a hypothesis. For example, when discussing a particular rectangle, the teacher–researcher asked the students to make a prediction about what would happen to the area if one changed the table of values such that the length and the width increased by 1 cm. Ally engaged in the generalizing action of searching (Ellis, 2007) for a pattern in the table. The searching action revealed a pattern that Ally described as a generalization: "Once you start going out it goes to 7. It . . . plus 7 squares, then there is a pattern. You just add 3 to every single one that you get." In her response to the prompt to make a prediction, Ally generalized that the rate of growth of the area would be 3 squares every time the length and the width each increased by 1 centimeter.

Encouraging sharing of a generalization or idea. Encouraging sharing of a generalization or idea includes formal and informal requests for sharing more broadly with the community as a whole. This served to shift a generalization or strategy from the private arena to the public arena so that other members of the community could consider, react to, and build on it. For instance, Daeshim worked with a table of values comparing the height and the area of a square ($y = x^2$). He

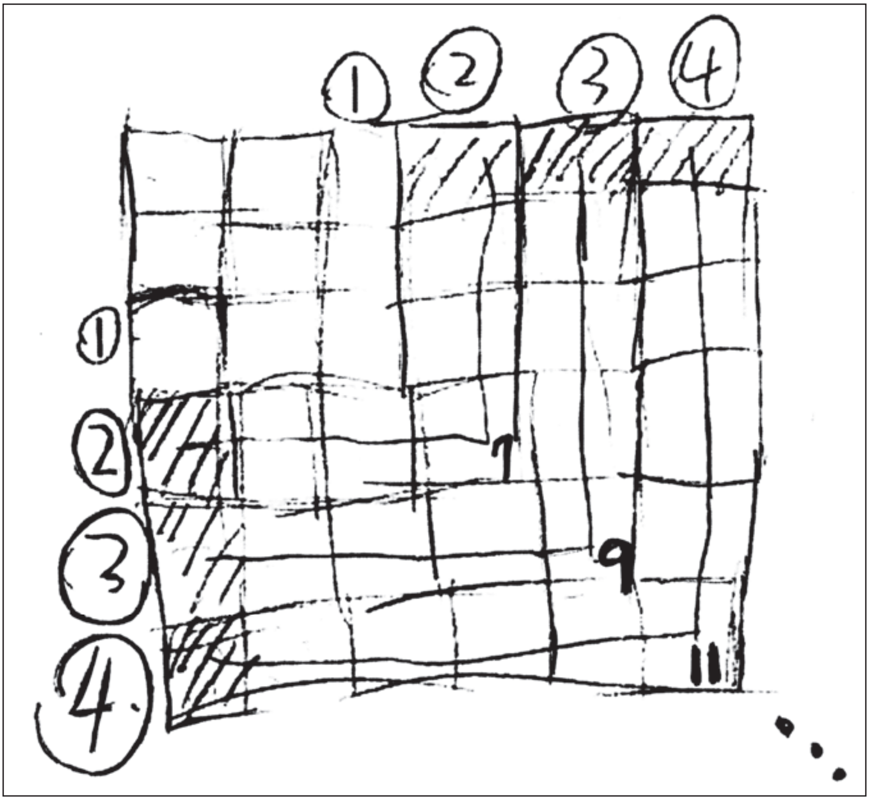


Figure 4. Daeshim's drawing of the first and second differences for a growing square.

engaged in the generalizing action of searching for a pattern by calculating the first and second differences between successive area values, finding that the second differences (DiRoG) were 2 cm^2 each time the height and length each grew by 1 cm. The teacher–researcher asked the students to draw a picture showing these differences, and Daeshim created the drawing shown in Figure 4. The teacher–researcher then asked him to share his drawing with the rest of the group. Daeshim placed on the board the drawing in Figure 4. As he wrote, Daeshim explained to the class, "7, it's the number of increase. Nine plus 7 is 16" [points to the smaller squares inside of the square]. By stating "9 plus 7 is 16," Daeshim referred to the 9 square units in the 3×3 square, and then the 7 additional square units that he added to it to make the 4×4 square with an area of 16 square units. The small numbers "7, 9, and 11" that Daeshim placed in his drawing indicate the 7, 9, and 11 additional square units one must add, respectively, when the 3×3 square grows to a 4×4 square, then to a 5×5 square, and finally to a 6×6 square. Reflecting on Daeshim's drawing, Bianca used the figure to come up with a new idea:

Bianca: So here we've got 1, 2, 3, 4, 5, 6 [counts the squares along the length of Daeshim's square]. So every time you increase the length and width by 1, you're increasing it

> by 2 [adds 2 new squares to the length and the width]. So, that's the 1 you're increasing it by height and then added to the 1 you're increasing by width is the 2. So, it can be anything [draws another square around the perimeter], along, it can be any 2 along these, it doesn't matter. But each time you grow the square by 1, the extra ones (meaning the additional squares, 7, 9, 11, . . .) grow by 2.

Bianca previously had developed a similar drawing showing the first and second differences for a growing square, and like Daeshim, she identified the DiRoG to be 2 cm^2. However, Bianca had identified the DiRoG to be two *specific* squares in her drawing, namely one square in the bottom right corner and one square in the top left corner. Although it is not possible to infer Bianca's thinking with certainty, it appears that observing Daeshim's drawing, with different squares shaded, helped her to realize that the 2 cm^2 DiRoG did not have to represent two specific squares, but instead could represent any two additional squares. In her explanation, Bianca used Daeshim's drawing to show that the DiRoG could be conceptualized as any two square units, but each time that difference must be 2 when the length and width each increase by 1 cm. This was a generalization that Bianca had not previously identified before considering Daeshim's drawing.

Publicly sharing a generalization or idea. Actions coded as publicly sharing involve directly shifting a generalization, drawing, strategy, or idea to the public arena in the form of revoicing (Forman, Larreamendy-Joerns, Stein, & Brown, 1998) or publicly validating or rejecting another member's generalization. For instance, the teacher–researcher revoiced 2 students' generalizations when she said, "So different patterns. One, Daeshim said that if the length is growing 1 cm, the area's growing 3 square cm. And Bianca said that area is 3 times the length every time." (As seen previously in the Encouraging Generalizing section, Ally also made a generalization similar to Daeshim's at a different time during the session.) The majority of the revoicing acts (all but two) were carried out by the teacher–researcher.

Members of the classroom also would publicly acknowledge a student's generalization as either correct or incorrect, thus drawing attention to it. There were times when a student rejected his or her classmate's generalization as incorrect, and in these cases his or her peers would respond by adjusting the generalization or creating a new one. Members of the classroom also publicly validated generalizations. For instance, in the dialogue that follows, the teacher–researcher validated a generalization, and the students responded by continuing the line of thought and creating more generalizations:

Sarah: I think 'cause the DiRoG you go up twice, it's squared.
Tai: [Gasps] you go up three times it's the . . . it's . . . oh, it's cubed!
Tai: It's like DiDiRoG!
TR: You . . . you're right. You're right.
Sarah: I'm right?
TR: And that works for cubed, it works,
Sarah: It works for the hundredth power?

In response to the public validation, Sarah engaged in the generalizing action of extending (Ellis, 2007) this pattern to ultimately create a general rule, concluding with the generalization that cubic functions would have constant third differences, quartic functions would have constant fourth differences, and nth-degree functions would have constant nth differences.

Encouraging justification or clarification. Encouraging justification or clarification actions include requests to explain or clarify a solution or pattern or prove a generalization. These actions encourage reflecting on the generalization or solution at hand, and these reflections can initiate examination of the general nature of the properties and relationships being discussed. For instance, the teacher–researcher's statements were coded as encouraging justification or clarification when she asked, "The conjecture is that it's 3. But why 3? What does the 3 have to do with the area?"

A participant may also ask another classroom member to explain how he or she came up with a particular result. This type of action was carried out most frequently by the students. For instance, Tai's actions were coded as encouraging clarification when he asked, "h times 6.75 times h, how'd you get 6.75?" Part II illustrates the ways in which the students' encouraging justification or clarification actions prompted a number of generalizing actions such as creating general rules, extending beyond the case at hand, and the development of general patterns.

The following excerpt exemplifies how a prompt to clarify resulted in the creation of a new generalization. The students worked with a 4×2.5 rectangle that grew proportionally by maintaining its height/length ratio and were trying to determine what the DiRoG would be for a table in which the height increased by 1 unit each time, although the students had not yet created an equation. The area can be represented by $y = 0.625x^2$, in which x is the height of the rectangle and y is its area. The DiRoG for a well-ordered table for a quadratic function $y = ax^2$ in which the height increases each time by 1 unit will be $2a$, or in this case, 1.25 cm^2 (see the Appendix). Jim explained that he could "reduce the height until 1 and then multiply the length by . . . so. Four divided by 4. And so, 2.5 divided by 4, times 2 equals 1.25." Jim found the rectangle's length when the height was 1 to be 2.5/4, which is 0.625, and he knew that the DiRoG would be twice this ratio. Then Ally asked Jim for clarification:

Ally: Wait, what did you do again? What did you do? You did 6? (*Encouraging Clarification*)

Bianca: I did 4,

Ally: What is your equation? It's height over, over width?

Bianca: Over 2.5 equals 2 over 1.5 . . . 1.25 . . .

Ally: Times width, times . . . height over width times what?

Bianca: Height over length times 2. So I guess it's like, you just do $2H$ over L.

Bianca answered Ally's question to Jim because Jim and Bianca had worked together before the group discussed their ideas as a whole and had developed a strategy for determining the DiRoG. She found the ratio of the rectangle's sides by reducing the height to 1 unit and finding the corresponding length. For instance, for any given rectangle with height H and length L, Bianca would create a fraction H/L, and then find an equivalent fraction with an H-value of 1. She would then take the denominator, L, and double it to find the DiRoG. Bianca eventually realized that instead of "reducing" the fraction until H was 1 and doubling the denominator, she could "reduce" the fraction until H was 2 and take the denominator value as the DiRoG. So for this problem, Bianca took the height/length ratio, 4/2.5, "reduced" it to 2/1.25, and circled the denominator, 1.25, as the DiRoG.

Ally's questioning caused Bianca to begin to redescribe her strategy with the numbers, but Ally interjected by asking Bianca to describe her strategy more generally by referencing the width and height: "Times width, times . . . height over width times what?" In response, Bianca shifted from the specific strategy she employed to describing a general strategy, "height over length times 2." This statement may have been a way to describe her strategy of reducing the H/L fraction until $H = 1$, and then multiplying the denominator, L, by 2. Bianca then may have formalized this further in her statement "I guess it's like, you just do $2H$ over L." It is also possible that "$2H$ over L" could have referred to Bianca's strategy of reducing the fraction until $H = 2$, and then taking the corresponding L-value as the DiRoG. In expressing her strategy this way, Bianca engaged in the generalizing action of extending her strategy to a general H by L rectangle. The students later created an alternate form of this generalization that the ratio of twice the length to the height of a rectangle will produce the second differences value.[4]

Building on an idea or a generalization. This action refers to instances in which a member of the community builds on another member's idea or conclusion. This can occur in a number of ways, including using another person's idea to create a new relationship or generalization, revising another member's strategy or generalization, or creating an artifact or problem statement in response to a member's idea, strategy, or generalization. It can also take the form of an extended period of argumentation between multiple students, who collaboratively reflect upon and refine a series of generalizations, as seen in Part II. However, teacher actions, including the development of new problems, can also either prompt building or engage in a form of building. For instance, consider the drawing that Daeshim (shown in Figure 4) created depicting the growth of a square. This drawing emerged from a problem the teacher–researcher wrote that referenced a student's table from the previous day (see Figure 5). The problem task made use of a student's prior creation to propose a new series of tasks, namely, drawing the growing square and showing the first and second differences in the drawing.

[4]Bianca's generalization and the alternate form $2L/H$ is correct for height/length/area tables in which the height values increase by 1, but it is not true for tables in which the constant increase is not 1 (see the Appendix).

Yesterday when you looked at the square that grows in both length and height, you made a table to compare the length/height and the area. Here is what one student's table looked like:

Length/height	Area		
3	9		
		+7	
4	16		+2
		+9	
5	25		+2
		+11	
6	36		+2
		+13	
7	49		+2
		+15	
8	64		+2
		+17	
9	81		

Make a drawing of the growing square:

- Start with a 3 × 3 square.
- Show how you increase to get a 4 × 4 square.
- Show how you increase to get a 5 × 5 square.
- Show how you increase to get a 6 × 6 square.

1. Show where the +7, +9, +11, +13, etc. is in your drawings.

2. Show where the +2 will be in your drawings.

Figure 5. Problem building on a student's previously constructed table.

Focusing attention on mathematical relationships. Focusing involves encouraging members to attend to a specific mathematical feature of a problem, idea, or representation. The most common type of focusing took the form of the teacher–researcher focusing students' attention on the quantities of length, height, and area embedded in the problem situation. For instance, after the students began developing generalizations about how to find the second differences from a table of values, the teacher–researcher focused the students' attention back on the original rectangle scenario, asking "Is the difference in the rate of growth in this case the length, or the area, or something else?" This prompted one of the students, Sarah, to reconceptualize her generalization in terms of the quantities of the rectangle, stating that the second differences could be found from "the length divided by the width multiplied by 2."

Part II: Interaction Cycles

The analysis revealed patterns of students creating and refining their generalizations through cyclical interactions, in which each round of generalizing prompted the development of new generalizations. This section presents two episodes that demonstrate the manner in which the students and the teacher–researcher together engaged in generalizing-promoting actions and then built on one another's ideas to develop more refined generalizations over time. Both episodes demonstrate actions observed in extended interaction cycles among students: (a) encouraging justification or clarification, (b) building on an idea or generalization, and (c) publicly generalizing.

The first episode highlights student–student interactions, in which the students worked together in a small group without the teacher–researcher's immediate intervention. The students' actions encouraged justification or clarification by focusing on the origins of one another's generalizations, and the students also repeatedly engaged in building actions as they responded to one another's ideas. The second episode highlights the teacher–researcher's actions. In this episode, the teacher–researcher also emphasized actions that encouraged justification or clarification, but she focused primarily on prompting students to explain why their generalizations and strategies made sense. These acts, in combination with publicly sharing students' generalizations, encouraged the students to publicly generalize, creating statements that the students could reflect on, build on, and refine. Taken together, the episodes demonstrate how both teachers and students can contribute to the development of positive interaction cycles that promote extended generalizing.

Excerpt 1: Developing an equation to find the DiRoG. On the 9th day of the teaching experiment, the students worked with the table in Figure 6, detailing the heights and areas of similar rectangles. The quadratic relationship depicted in the

Here is a table for the height (in cm) versus the AREA (in square cm) of a rectangle that is growing in proportion:

Height	Area
5	75
10	300
15	675
20	1,200
25	1,875
74.2	?

Figure 6. Table of height and area values for a proportionally growing rectangle.

table is $A = 3h^2$, which the students quickly determined. The first differences between the successive area values are not constant, but the second differences for this table can be calculated as 150 cm^2 for each 5-cm increase in height. The students had been exploring how different table configurations for the same function, in which the height increased by constant values of 1, 2, or 3 cm, resulted in changes in the second differences. They had never encountered a table in which the Δh value was 5 cm and were interested in trying to predict what the second differences would be.

One student, Daeshim, predicted that the second differences (the DiRoG) would be 750. Bianca engaged in the act of *encouraging justification or clarification* when she asked, "So how did you get that, Daeshim?" Daeshim explained his thinking:

Daeshim: 5 squared times 2 times 15.
Jim: I'll just show you the DiRoG. It's not 750! (*Publicly Sharing; Rejecting*)
Bianca: Where'd you get 15? (*Encouraging Clarification*)
Ally: It's not 750.
Daeshim: 15 is like . . .
Jim: It's not 750. The DiRoG's not 750. That's too high.

Height		Area
5	15	75
10	30	300
15	45	675
20	60	1,200
25	75	1,875
74.2		?

Figure 7. Daeshim's additional length column for the height/area table.

Bianca highlighted Daeshim's generalization by asking him to explain its origins, and Jim and the others further directed attention to his generalization by rejecting its correctness. In creating his generalization, Daeshim had introduced a third column, between the height and area columns, depicting the length of each rectangle (see Figure 7). He noted that the difference between successive length values was 15 cm, and Daeshim used a prior generalization, which had previously been established and made public, to calculate the second differences: "*DiRoG* = $2 \cdot \Delta h^2 \cdot \Delta l$."[5] Thus, Daeshim calculated 2 • 52 • 15 = 750. This generalization was correct when the difference between successive height values was 1, but it was not correct for

[5]This prior generalization is in error by a factor of Δh (see Appendix for further discussion).

other Δh values. In contrast, Jim calculated the DiRoG directly from the table. Bianca built on both Daeshim's original idea and Jim's result to state:

Bianca: 750 divided by 5 is 150.
Jim: That's, that's the DiRoG!
Bianca: Yeah.
Jim: Because we're going up by 5s and you have to divide it by 5? (*Publicly Generalizing*)

Bianca's actions in building on Daeshim's and Jim's work supported Jim's generalization. She then built on Daeshim's original prediction and Jim's calculation, publicly generalizing by attempting to make a connection between the two. Bianca asked aloud, "How are these related? I'm just trying to figure out a general equation." Tai responded with "The difference in the height . . .," which encouraged Bianca to attend to that difference, which the students called *DiH*: "Oh! It's like, it's 5 squared times 2 times 15 over *DiH*!" (*Publicly Generalizing*)

The process of reacting to and building on Daeshim's and Jim's predictions and public generalizations supported Bianca's development of a new general rule that the *DiRoG* is equal to 5 squared (i.e., Δh^2, or as Bianca would describe it, DiH^2) times 2 times 15 (i.e., Δl, or *DiL*) divided by Δh, (or *DiH*). Bianca's reasoning corrected for the original generalization Daeshim had used in order to account for the differences in successive height values being 5. However, it was not clear to Bianca at this point whether her equation was specific to this particular table or could be generalized to other uniformly increasing quadratic tables (see the Appendix for an explanation of why this equation is correct for any table of $y = ax^2$ data). Jim's questioning then led to a refinement of this generalization when he asked, "Where'd you get this 'times 2'?" (*Encouraging Justification or Clarification*). Bianca responded, "It's like, it's just the equation. It's how it works. So it's 5 squared times 2 times 15 over *DiH*." Jim pursued this, asking for another clarification: "Where'd you get the 15?" This caused Tai to realize, "Oh, it's the, it's length!" As the students continued to talk, Jim then realized, "Oh, it doubled! It's doubled! That's where the 2 comes from, it's doubling the length!" Building on Jim's doubling idea, Bianca then said, "*DiH* squared times 30 over *DiH*." (*Publicly Generalizing*)

Bianca was not sure whether her generalization truly held for different Δh values, or whether it was only particular to this table. Daeshim engaged in the generalizing action of relating Bianca's equation to his own way of representing the DiRoG, and stated, "It is original height and the original length. So, 2, original height squared times 2 times original length divided by original height" [writes $DiRoG = 2(O.H.)^2(O.L.)/(O.H.)$].[6] "It is the same thing" [points to Bianca's equation]. Daeshim's equation relied on the first table entry for height and length, but he acknowledged that it was the same as Bianca's idea. The students eventually decided to rely on the change in height and length values rather than the first table entry values

[6]Daeshim's version is correct only for tables in which the height is increased by the original value; for example, for an $m \times n$ rectangle, the height must increase by m each time. See Appendix for a discussion of this generalization.

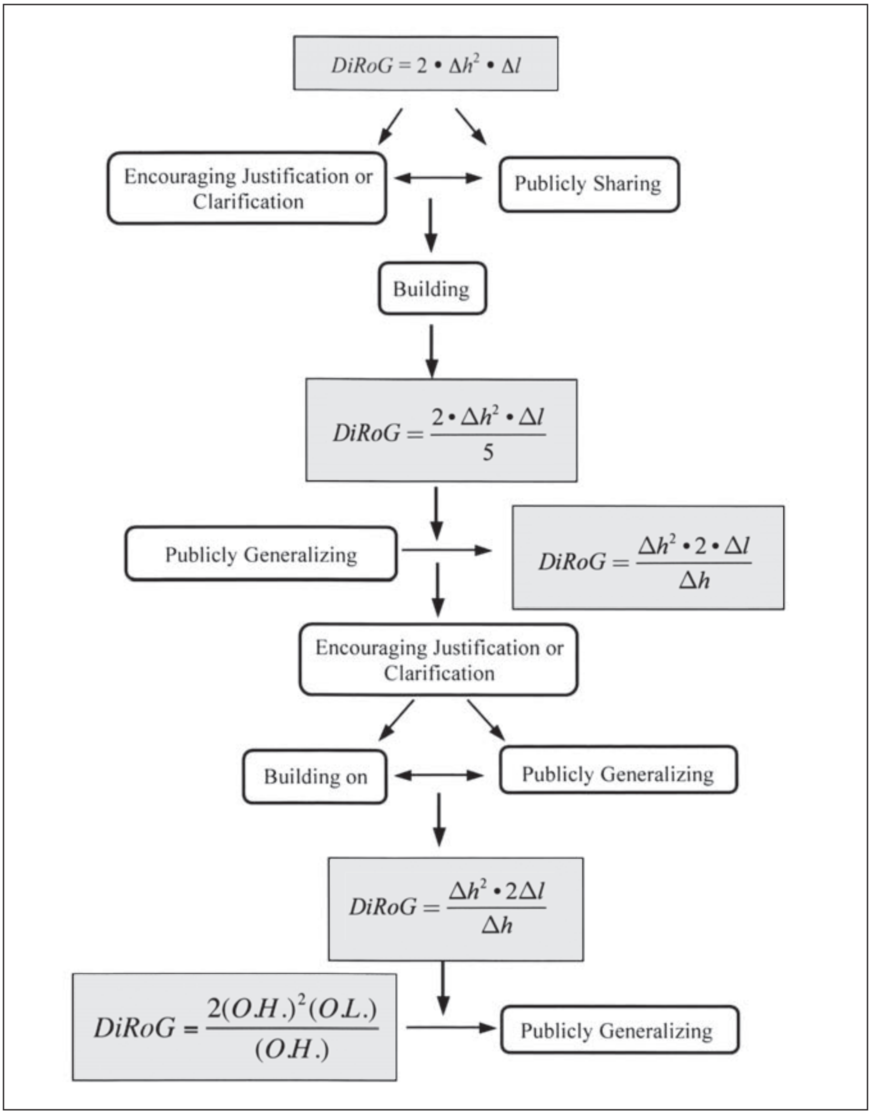


Figure 8. Generalizing-promoting actions and their associated generalizations.

and ultimately formalized their generalization as "$DiRoG = 2 \bullet DiH^2 \bullet DiL/DiH$," which provides a correct way to determine the DiRoG for any uniformly increasing quadratic table of values representing the growing rectangle scenario.

Figure 8 depicts the process of generalizing and engaging in generalizing-promoting actions described in Excerpt 1. The original problem statement served as an initial launching point for the students' investigation, but the students quickly turned the problem into a new question to address that about which they were curious, namely, how to predict the DiRoG of the area from any well-ordered table of height/length/area values. They began with an existing generalization of Daeshim's, and the students' process of encouraging reflection by questioning the origins of Daeshim's generalization fostered the refinement of his equation in order to account for a new table with a Δh value of 5. As seen in Figure 8, the students' collaborative process of encouraging reflection, building, and publicly generalizing supported the development and continual refinement of an equation for finding the DiRoG from a height/length/area table. The shaded rectangles represent the generalization that evolved over the course of the students' conversation, whereas the rounded rectangles represent the students' generalizing-promoting actions. It is particularly notable that the three actions of encouraging justification or clarification, building on an idea or generalization, and publicly generalizing recurred over the course of the episode, even as the specific content of those actions changed throughout the course of the conversation.

Excerpt 2: Connecting the DiRoG to the height and length of the rectangle. After having determined ways to identify the DiRoG for well-ordered tables, the students began to encounter tables of data that were not well ordered. The teacher–researcher introduced tables that were both linear and quadratic, linking to the classroom history that identified the quadratic tables with rectangles that grew by maintaining their length/height ratios and linking the linear tables to rectangles that grew in one direction only. The students encountered the "mystery" table shown in Figure 9.

Here is a table for a rectangle that's growing in a way I'm keeping secret:

Height (cm)	Area (cm^2)
2	1
5	6.25
7	12.25
8	16
10	25
1	_____
1/2	_____
h	_____

1. Could this rectangle be growing in one direction only, or in both directions? How do you know?

2. What type of graph do you think this will be? Make a prediction:

Figure 9. "Mystery" table of quadratic data.

Even though many different types of rectangles could generate the values in the table in Figure 9, the students constrained the problem to two options (a rectangle growing proportionally in both directions versus a rectangle growing only by height, keeping the length constant) due to their prior experiences in the classroom. (The data can be represented by the function $y = 0.25x^2$.) One student, Sarah, told the class that she had added a middle column for length, with values 0.5, 1.25, 1.75, 2, and 2.25. The teacher–researcher encouraged justification or clarification by asking the students to consider why the length was important in answering the questions:

TR: So why does finding the length help? What's it tell us?
Jim: Between 8 and 7, which is the one that I found the easiest, it went up by .25.
TR: Uh huh.
Jim: And then I kinda worked backwards.
TR: And why'd you pick between 8 and 7? (*Encouraging Justification or Clarification*)
Jim: Because they were going up by 1s.

Having reflected on the utility of increasing by 1s, Jim used this information in a building action to re-create a table in which the height values increased by 1 and the length values increased by 0.25. He could then more easily calculate the differences between successive length and area values. The teacher–researcher again encouraged justification or clarification by asking the students to consider the purpose of calculating the difference between successive length values:

TR: So what does this .25 tell us, if anything?
Jim: It's the DiH.
TR: Difference in height? (*Publicly Sharing by Revoicing*)
Jim: Oh yes . . . no, DiL.
TR: DiL. Difference in length? (*Publicly Sharing by Revoicing*)
Jim: Difference in length times 2 . . . (Beginning to state a *Public Generalization*)
Bianca: Maybe is the, is the DiRoG of the area over 2 equal the DiL of the length?

Jim's public generalizing appeared to prompt a new potential generalization for Bianca, in which she hypothesized that the DiRoG over 2 is equal to the difference in the length. Bianca simply may have noticed that this was the case for the re-created table, because the difference in successive length values was half the DiRoG. Bianca's hypothesis is correct: The function represented in the table is $A = 0.25h^2$. This is true for tables in which the height value increases by 1, because the value a in $y = ax^2$ is equivalent to the change in length, and the second differences can be calculated as $2a$ for any $y = ax^2$ table in which Δh is 1. However, it is not clear whether Bianca thought of her generalization as universal or as pertaining to this particular table only.

The teacher–researcher again encouraged the students to reflect on the difference in lengths:

TR: So, how does this help us figure out our equation for area? (*Encouraging Justification or Clarification*)

Bianca: Well, if you use all that criteria, you're bound to figure out something.

TR: [Laughs] Yeah. That's true. You'll have to figure out something.

Jim: 4 times . . . 4 times . . . 2.5 is your length. And then you do it again, which is 2.5 *h* squared. (*Publicly Generalizing*)

TR: So, 2.5 *h* squared? (*Publicly Sharing by Revoicing*)

Jim: Yes.

Bianca: I bet you . . .

Jim: Oh, sorry, sorry! .25 *h* squared.

TR: Oh, .25 *h* squared? (*Publicly Sharing by Revoicing*)

Jim: Yes.

Bianca: Wait! I bet that we'll find, or possibly we could find, some one equation that you could just plug everything into, and it would work.

The teacher–researcher's repeated prompts for the students to reflect on the purpose of calculating the difference in length values encouraged Jim to develop a generalization, which he stated incorrectly. Her revoicing actions then prompted Jim to revise his generalization. The teacher–researcher was unsure whether all of the students followed Jim's reasoning, particularly because his utterances as he thought out loud were not particularly clear. Therefore, she once again encouraged justification or clarification, this time asking about the origins of Jim's generalization:

TR: I'm kind of curious, how'd you figure this out, though? The .25 *h* squared equals area. [Tai raises his hand.] Tai?

Tai: Well, we kind of . . . well, I kind of looked at the 7 and 8. Because they were like, closer. And then, well, like the number in the front [points to the 0.25 in $0.25h^2$], is always, like, the difference in the length divided by the difference in the height.

In response to the teacher–researcher's prompts, Tai stated another generalization, that the a in $y = ax^2$ is the ratio of the change in length to the change in height for a rectangle growing in proportion to itself. The teacher–researcher publicly shared Tai's contribution by writing it on the board: "*DiL*/*DiH* = # in front of h^2." This act prompted the following response from Bianca, who said, "Wait, but isn't that the same thing as DiRoG/2 equals . . . ?" The teacher–researcher validated Bianca's attempt by stating "It is," and then Ally built on Bianca's remark to exclaim, "So that means that *DiRoG* over 2 equals *DiL* over *DiH*!"[7]

Refining generalizations. In contrast to the first episode, this episode is driven by the teacher–researcher's guiding remarks as the group discussed ideas as a whole. Figure 10 depicts the relationships among the generalizing-promoting actions, strategies, and generalizations that occurred in the episode. The rounded rectangles represent the generalizing-promoting actions—those carried out by the teacher are dashed. The generalizations are shaded, and the strategies students used are depicted in ovals. At times, two actions appeared to work together to promote

[7]This is only the case for this specific table that the students rewrote so that the increase in height values was 1; it is not valid for every table (see Appendix).

a subsequent generalizing action or other action. For instance, toward the end of the episode, the teacher–researcher's act of publicly sharing Tai's generalization appeared to support Bianca's action of publicly generalizing, and these actions worked together to foster Ally's building action to create her new generalization.

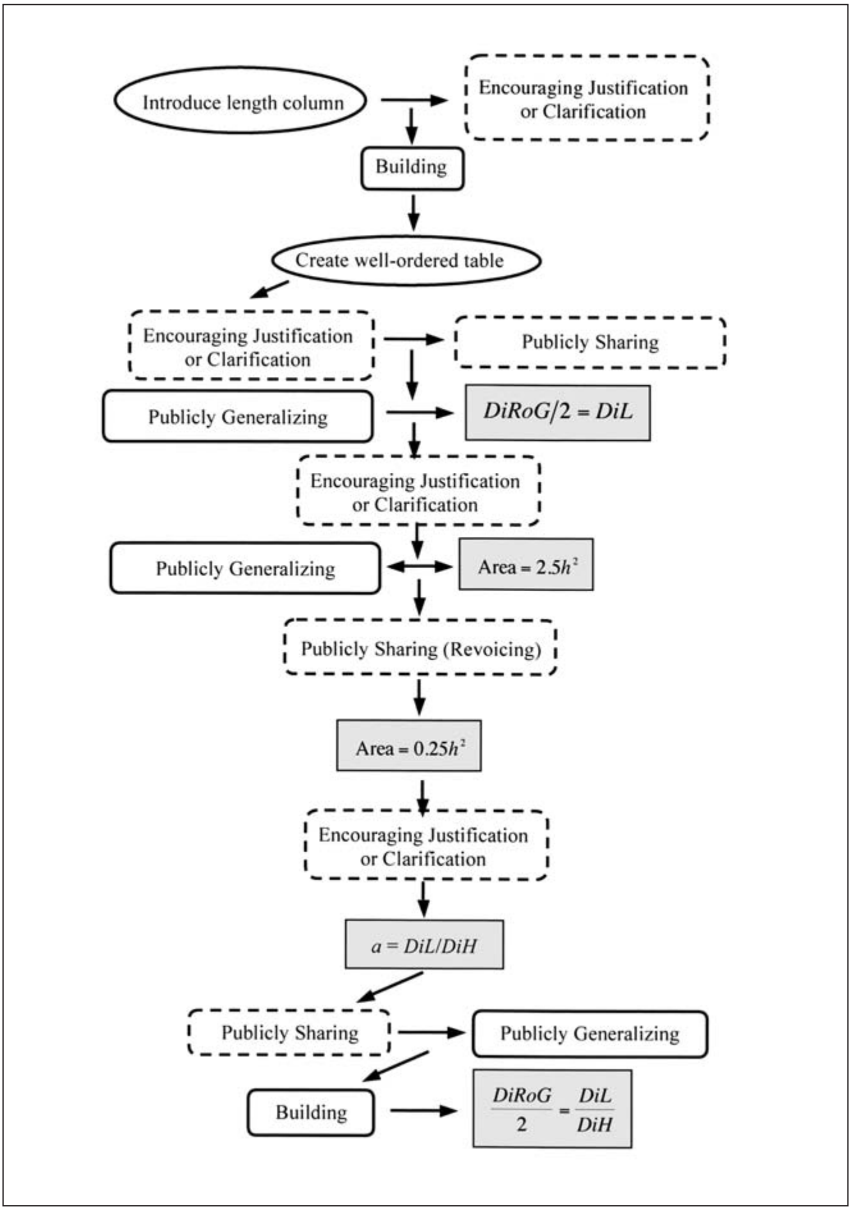


Figure 10. Relationships among generalizing-promoting actions and generalizations.

Throughout this episode the teacher–researcher encouraged the students to reflect on the generalizations they were developing, particularly to explain why their strategies and ideas made sense. The initial strategies on which the students relied, such as introducing a third column (length) to the table and creating a new well-ordered table, also worked together with the generalizing-promoting actions to advance the students' thinking. The students' responses to encouragement to reflect on their strategies and generalizations, in combination with the teacher–researcher's revoicing acts, supported the development and refinement of new generalizations. As seen in Figure 10, it was a combination of the students' strategies, the teacher–researcher's actions, and the students' actions that seemed to foster the evolution of the generalizations that occurred throughout the episode.

Students' ownership. There were times when making and encouraging generalizations required the teacher–researcher's explicit support. At other times, as seen in Excerpt 1, the students took up this role themselves. Students' ownership of the encouraging roles can also be seen as they encouraged justification and clarification. The teacher–researcher's statements encouraged the students to explain why their strategies and generalizations made sense, which contributed to a classroom culture that emphasized explaining why. The students also jointly contributed to this culture, and their ownership of it is seen throughout Excerpt 1 as they repeatedly asked one another to explain or justify their statements.

The nature of the problems presented and the sense of ownership that the students took over them also contributed to the chain of generalizing. The problem in Excerpt 1 required students to determine an area value, but it was the organization of the table in 5-cm increments for the height that sparked the students' curiosity and encouraged their initial generalization attempts. Similarly, in Excerpt 2, the students' interaction with the problem situation encouraged generalizing in a particular direction. They adapted the problem by creating new tables, which encouraged the development of new generalizations that attended to the differences in both height and length values. The students' curiosity about particular features of the various problem contexts was influenced by the prior generalizations they had developed, and also simultaneously influenced the subsequent generalizing in which they engaged.

DISCUSSION

Generalization is acknowledged to be a critical component of mathematical activity but remains difficult for students to do successfully and for teachers to support effectively (Blanton & Kaput, 2002; Jurow, 2004; Lannin, 2005; Mason, 1996). To better understand the types of interactions that support students' generalizing, this study examined the mathematical activity of 6 students and their teacher–researcher as they explored quantitatively rich situations about quadratic growth in the form $y = ax^2$. In viewing the generalizing-promoting actions and the interaction cycles together, a number of specific actions emerge as particularly salient in their ability to foster generalizing. The actions of (a) encouraging justifying and clarifying; (b) publicly sharing students' contributions, particularly through revoicing; and (c) encouraging generalizing, especially through prediction activities, are powerful classroom actions that can support the generalizing process, as has been suggested by other work (e.g., Cobb et al., 1997; Hall & Rubin, 1998; Jurow, 2004; Lehrer, Strom, & Confrey, 2002; O'Connor & Michaels, 1996). In addition to actions identified in prior research, new actions also emerged as playing a key role in supporting students' generalizing: Actions such as (a) building on an idea or generalization, (b) publicly generalizing, and (c) focusing attention on mathematical relationships were important in fostering the generalizing process.

Generalizing as Situated

This study continues the tradition of considering how generalizing operates as a situated activity. Learning arises out of collective representations that are rooted in a community, occurring through experiences that are mediated by interaction, language, and tools; the same is true of generalizing. One intention of the teaching experiment was to emphasize the nature of one type of quadratic growth through the relationship between the height, length, and area of growing rectangles, but the students co-opted these problems in order to investigate what the constant second differences represented in tables of quadratic data, how the second differences were related to the functional formulas relating height and area, and how to predict the second differences for different table configurations. The robust nature of the rectangle context and its associated tasks supported these investigations and thus was an important factor in influencing the types of generalizations the students produced.

The students' control over the direction of their mathematical investigations required a continual revision of the hypothetical learning trajectory guiding the initial development of the teaching experiment, as well as the nature of the problems the research team developed on a daily basis in response to the students' evolving interests. The activities changed from emphasizing direct functional relationships between the height and area of rectangles to ones that promoted a deeper understanding of (a) what the DiRoG represented in terms of the areas of the rectangles, (b) how the DiRoG could be predicted for a well-ordered table of data with any Δh value, (c) how the DiRoG was connected to the parameter a for $y = ax^2$, and (d) what the value of a represented in terms of the height and length of a rectangle.

The students' investigations of these questions promoted actions such as publicly generalizing, publicly sharing, building, and encouraging justification or clarification as they attempted to make sense of a new mathematical domain. These actions also shaped the generalizations they produced. Engle (2006) articulated a situative theory of transfer, in which she analyzed learners' participation in constructing the content that they could transfer to new contexts. The interaction cycles similarly identified ways in which the students participated in shaping the direction of the generalizations that they ultimately developed. From the interactionist perspective, this underscores the notion that mathematical themes are not fixed but are instead interactively constituted and change through the negotiation of meaning (Herbel-Eisenmann, 2003).

Nemirovsky (2002) described the notion of *situated generalization*, attending to the lack of a true separation between generalizing and the realm in which it takes place, and Goldstone and Wilensky (2008) described a *grounded interpretation* account of generalization, in which transportable understandings come from the interaction between the physical elements of simulations and the interpretations of those elements. Similar to the way in which others have described situated generalization (e.g., Carraher, Nemirovsky, & Schliemann, 1995), Goldstone and Wilensky emphasized the notion that concrete and abstract learning are interconnected. The results from this study mirror these stances, highlighting how students built on one another's drawings, representations, and ideas to create generalizations that articulated new relationships and yet remained connected to the artifacts that fostered them. This can be seen in Bianca's use of Daeshim's drawing of a growing square to explain that the DiRoG must always be 2 squares, as well as in the generalizations that the students created that were dependent upon the particular height/length/area tables they developed, relating the DiRoG to the change in height and change in length values.

The notions of situated abstraction and abstraction in context also highlight the central role of mediating tools (Hershkowitz, Schwarz, & Dreyfus, 2001; Hoyles, Noss, & Pozzi, 2001; Noss & Hoyles, 1996). The generalizing activities of the students in the teaching experiment were tied not only to the problems, drawings, and tables they developed but also were shaped by the students' co-constructed use of language as they thought about height, length, area, and constant differences. The use of the dynamic software The Geometer's Sketchpad also influenced students' use of language, representation, and generalizing activities. Studies support the notion that students' reflection on the results they observe when working with dynamic software can influence the direction of their mathematical investigations (Hollebrands, 2007), and several researchers have investigated the various affordances of dynamic software that may support students' reasoning (Jones, 2000; Mariotti & Bartolini Bussi, 1998; Talmon & Yerushalmy, 2004).

In this case, the students' ability to easily compare changing height, length, and area values contributed to the nature of the tables they invented to organize and record these data. In particular, the fact that their dynamic explorations allowed for the adjustment of either height or length encouraged the students to include both quantities in tables comparing a side length to area, which then spurred the creation of generalizations that gave equal consideration to height and length, even though standard textbook single-variable quadratic functions representing area typically rely only on one or the other, such as $A = (1/2)h^2$. The language that the students created to account for the phenomena they noticed, such as *DiRoG*, *DiH*, and *DiL*, also shaped the direction of their generalizing. Many of the general statements that the students ultimately created, such as $DiL/DiH = a$, are intimately connected to the dynamic representation in Sketchpad, the students' resulting table configurations, and the specialized language developed in this particular setting, but they are also generalized representations of these relationships.

Generalizing as a Collective Activity

This study has described how the students' interactions supported and shaped their generalizing activities. The participants made decisions about what they valued and were interested in pursuing, which influenced the direction of their generalizing. The teacher–researcher also strongly influenced these activities, and the nature of this influence is particularly evident in the second interaction episode. The teacher–researcher's continued emphasis on publicly sharing generalizations, encouraging generalizing, and encouraging justifying and clarifying set the tone for a classroom culture that supported and encouraged these activities. As Bauersfeld (1995) noted when discussing the role of interactionism, "Social interaction in the classroom appears as teacher–student interaction as well as student-student interaction" (p. 153).

The interaction cycles emphasized the ways in which an initial generalization can evolve over the course of an extended period of interaction and reflection, passing through many different forms, to the point at which the final, stabilized version of the generalization cannot be said to have been developed by any one student in isolation of the group's interactions. This can be seen as *collective generalizing*, in the same way that Blanton and Stylianou (2010) documented processes of *collective proving*, and Cobb (2005) articulated the notion of *collective abstraction*. In each case, the collective activity, be it generalizing, proving, or abstracting, has its origins in the social plane of public discourse.

Learning from the possible: Implications. The study reported in this article was from a small-scale setting with 6 students and 1 teacher–researcher. Unlike a typical whole-class setting, here the teacher–researcher had the freedom to follow the students' mathematical interests and investigations without the constraints of time pressure or content-coverage pressure. Although this setting differed in significant ways from typical mathematics classrooms in the United States, it still provides an important lens on the types of interactions that are possible as students investigate mathematical contexts and develop generalizations. This work builds on Shulman's (1983) identification of the value of good cases in order to learn from the possible. From investigating this small-scale setting in which students collaborated in interaction to generalize together, it was possible to identify a set of actions that can support productive generalizing. The outcomes of this study suggest a professional development model that would foster teachers' abilities to (a) encourage justifying and clarifying, (b) publicly share students' contributions, (c) explicitly encourage generalizing, and (d) refocus students' attention on mathematical relationships. The generalizing-promoting actions, although they occurred in an idealized setting, are ones in which teachers and students could potentially engage within the context of whole-class settings. A necessary line of future work will entail the consideration of ways in which these actions could translate to larger classrooms and teacher education initiatives.

Jurow (2004) noted the importance of orienting and guiding students to the mathematically relevant properties in and across situations. Pedagogical moves in a whole-classroom context such as publicly sharing generalizations, encouraging generalizing, and encouraging justification and clarification can serve this role as students struggle to make sense of problems and contexts. The results of this study also suggest, however, that the role of an orienting guide need not be limited to the teacher; students who work in collaboration with one another also can play this role. The act of publicly sharing generalizations and conjectures opens a space for students to respond to, accept or reject, refine, and build on initial attempts in a process that sifts through multiple ideas in order to ultimately highlight those that are mathematically powerful. Activity-oriented classrooms that support students working together as they engage in mathematical generalizations could provide a fruitful setting to foster students' engagement in the types of generalizing-promoting actions identified in this study.

The results from this study contribute to a view of generalizing as a dynamic, evolving, and collective process, furthering the tradition of moving beyond viewing generalizing as an individual, cognitive construct (Davydov, 1990; Jurow, 2004; Reid, 2002; Tuomi-Gröhn & Engeström, 2003). By examining the generalizing-promoting actions as both discrete acts and as pieces of larger classroom interaction cycles, this study situated generalizing within the social context of the class setting, rather than focusing solely on the students' actions or the teacher's pedagogical moves. Instead, it provides a lens for viewing generalizing as a situated act that is influenced by—and influences—the interrelated actions of students, teachers, problems, representations, and artifacts.

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Accepted January 31, 2011

APPENDIX

The Accuracy of Students' Generalizations in the Growing Rectangle Context

The students in the teaching experiment worked exclusively with the growing rectangle context. The majority of their work (and all of the generalizations reported in this paper) was limited to the $y = ax^2$ case, as a representation of a rectangle that grows while maintaining its length/height ratio, as shown in Figure A1. Consider a specific case in which the original rectangle in Figure A1 is 2 cm high by 4 cm long. If we increase the height by 1 cm to become 3 cm, then the new length will be $4 + (4/2)(1) = 6$ cm. If we increase the height by 5 cm to become 7 cm, the new length will be $4 + (4/2)(5) = 14$ cm. If we increase the height by k cm to become $2 + k$ cm, the new length will be $4 + (4/2)(k) = 4 + 2k$ cm. In each case, regardless of the value of k, the length/height ratio of 2 is maintained.

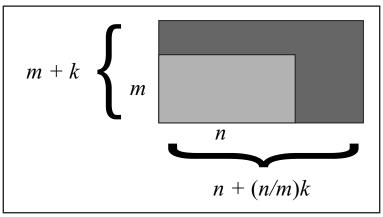

Figure A1. Rectangle that grows proportionally by maintaining its length/height ratio.

The students represented this growth in height/area tables and height/length/area tables. A general height/length/area table that represents this type of growth, in which the height increases by a uniform amount—some constant k—appears in Figure A2. Textbook treatments of quadratic functions (e.g., Murdock, Kamischke, & Kamischke, 2004) often refer to the *first differences* and the *second differences* in a well-ordered table of values of a polynomial function, and students are instructed that, for a polynomial function, constant first differences mean the function is linear, constant second differences mean the function is quadratic, and so forth. If we calculate the first differences of the area in Figure A2, we obtain

Height	Length	Area
m	n	mn
$m + k$	$n + (n/m)k$	$mn + 2nk + (n/m)k^2$
$m + 2k$	$n + (n/m)2k$	$mn + 4nk + 4(n/m)k^2$
$m + 3k$	$n + (n/m)3k$	$mn + 6nk + 9(n/m)k^2$

Figure A2. Height/length/area table in which the height grows uniformly by k.

$2nk + (n/m)k^2$, $2nk + 3(n/m)k^2$, and $2nk + 5(n/m)k^2$, respectively. Calculating the second differences yields a constant second difference of $2(n/m)k^2$ for every k-unit increase in the height.

To return to our specific case with a 2 cm by 4 cm rectangle, consider the two tables in Figure A3, in which the height values increase by uniform amounts. In Table 1, the height increases by 1 cm, and in Table 2, the height increases by 5 cm. For Table 1, the second differences can be directly calculated as 4 cm^2 for each 1-cm increase in height, and for Table 2, the second differences are 100 cm^2 for each 5-cm increase in height. Based on the results from the general table in Figure A2, we could also calculate the second differences directly with the formula $2(n/m)k^2$ for every k-unit increase in the height. Both tables refer to a rectangle in which the height, m, is 2 cm and the length, n, is 4 cm. Therefore, for Table 1 with $k = 1$, the formula yields $2(4/2)1^2 = 4$, and for Table 2 with $k = 5$, this yields $2(4/2)5^2 = 100$.

Table 1				Table 2		
Height	Length	Area		Height	Length	Area
2	4	8		2	4	8
3	6	18		7	14	98
4	8	32		12	24	288
5	10	50		17	34	578

Figure A3. Two height/length/area tables for a growing 2 cm by 4 cm rectangle.

If we consider m and n to be particular constants representing the height and the length of a rectangle, respectively, we can represent the area of the rectangle in terms of its height, h, as follows: $Area = (n/m)h^2$. So, for instance, for the 2 cm by 4 cm rectangle, we could express the area as $Area = (4/2)h^2$, or $Area = 2h^2$. When the height is 2 cm, the area of the rectangle is 8 cm^2. When the height is 17 cm, the area is 578 cm^2. This representation gives new meaning to the parameter a in $y = ax^2$: In this case, because we can express the area as $(n/m)h^2$, $a = n/m$, or the length/height ratio of the rectangle.

Notice also that because we calculated the second differences to be $2(n/m)k^2$ for each k-unit increase in the height, for a table in which the uniform increase in height values is 1 cm, the second differences are simply $2(n/m)$, or $2a$. However, for tables in which the uniform increase in height values is something other than 1, we must adjust by multiplying $2a$ by the square of the constant increase of the height value.

Translating the Students' Mathematics

The students created a number of different generalizations that they expressed in terms of formulas that relied on their invented terms. Some of these generalizations were correct only for a particular type of table or for a limited domain, whereas others were more broadly applicable. This section assesses the correctness of the students' generalizations in comparison to the mathematics discussed in relationship to the general rectangle in Figure A1 and the general table in Figure A2.

In the Publicly Generalizing section, there is a quote from Bianca, who states, "Isn't $DiL/DiH = a$ the same thing as mine, as the *DiRoG* over 2 equals a?" What do these two formulas mean? *DiL* represents the uniform increase in length values in a height/length/area table, and *DiH* represents the uniform increase in height values. Is the first statement $DiL/DiH = a$ true for any table in which the height values increase by some uniform amount k? Considering this in relationship to the general table in Figure A2, *DiL* is $(n/m)k$, and *DiH* is k. Therefore $DiL/DiH = (n/m)k/k = (n/m)$. We have already established that $(n/m) = a$, so the generalization $DiL/DiH = a$ is correct. This makes sense, because we can also think of DiL/DiH in terms of the rectangle: *DiL* represents how much the length grows when the height increases by *DiH*, and the ratio of the increase of the length to the increase in the height will be equal to the length/height ratio of the rectangle, that is, a.

What about the statement that $DiRoG/2 = a$? Here *DiRoG* is the "difference in the rate of growth" of the area, or what I referred to previously as the second difference. Is this statement true for any uniformly increasing table? The second difference (*DiRoG*) for Figure A2 is $2(n/m)k^2$. Dividing this by 2 yields $(n/m)k^2$. Because $a = (n/m)$, Bianca's generalization is not true for any table: It only holds for tables in which $k = 1$.

Bianca makes another generalization that is reported in the Encouraging Justification or Clarification section, and an alternate form is referenced by Sarah in that section. Bianca's generalization, "height over length times 2" or "$2H$ over L," referred to a strategy in which she took the original height and length of the rectangle, expressed their ratio as a fraction H/L, and either simplified the fraction until $H = 1$ and doubled the length to find the DiRoG or reduced the fraction until H = 2 and used the corresponding length value as the DiRoG. This generalization was based on a table in which the height increased by 1-unit values. Using the language from Figures A1 and A2, this would be equivalent to forming the fraction m/n and converting it to an equivalent fraction in which the numerator is 1, that is, $1/(n/m)$. Then one would take twice the value of the denominator as the second differences, that is, $2(n/m)$. However, for any table with a uniform increase of k units for the height, the second differences will be $2(n/m)k^2$. Therefore, Bianca's generalization is appropriate for the specific table with which the students were working, because in that case k was 1. However, her generalization is not generalizable to any uniformly increasing table of quadratic data. Similarly, Sarah's remark that the second differences could be found from "the length divided by the width multiplied by 2" would be equivalent to calculating $2(n/m)$ as the second differences, which again would be in error by a factor of k^2.

In Excerpt 1 of the Results Part II, recall that the students worked with a table in which the height increased by 5 cm, and the students had already correctly determined the function to be $A = 3h^2$. The excerpt begins when Daeshim uses a previously developed generalization, $DiRoG = 2 \cdot \Delta h^2 \cdot \Delta l$. Translating that to the language in the general table in Figure A2, the term Δh is the value by which the height increases, or k. The term Δl is the value by which the length increases, or $(n/m)k$. Therefore, Daeshim's original generalization can be expressed as $DiRoG = 2k^2(n/m)k$, or $2k^3(n/m)$, instead of the correct formula for calculating the second differences, $2k^2(n/m)$. This is why Daeshim's original calculation was in error by a factor of 5 and was 750 cm^2 instead of 150 cm^2. For previous tables in which the increase in height values was 1, this did not matter. Jim's statement that "you have to divide it by 5" is correct.

Bianca then built off of what Daeshim had developed to conclude that $DiRoG = (DiH^2 \times 2 \times DiL)/DiH$. Expressed in the language of the general table in Figure A2, this can be expressed as $DiRoG = k^2 \times 2 \times (n/m)k/k$, or $2(n/m)k^2$. Therefore, Bianca correctly adjusted Daeshim's generalization to create a way to calculate the DiRoG for any uniformly increasing table. Daeshim also expressed this idea as $2(O.H.)^2(O.L.)/(O.H.)$, indicating "It is the same thing," where *O.H.* refers to the original height of the rectangle (m) and *O.L.* refers to the rectangle's original length (n). Daeshim's representation can be rewritten as $2m^2n/m$, or $2mn$. Recall that the way to determine the second differences in a general table is to calculate $2(n/m)k^2$, so Daeshim's generalization will only hold when $k^2 = m^2$; in other words, this is only true when one creates a table in which the height increases by iterating itself, so that the height values are m, $2m$, $3m$, and so on.