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The Proof Is in the Practice

Imagine an eighth-grade classroom where students are exploring relationships between the heights and areas of rectangles (Ellis 2011). They are investigating a special set of rectangles that grow in a particular way by iterating the height and length values so that the rectangles remain similar. One set of these rectangles is seen in **figure 1**. The students create tables, comparing the rectangles' heights and areas, and notice something interesting: The second differences for the area are always constant, no matter what type of rectangle the students use (see **fig. 2**). One student, Sara, makes the conjecture that this will be the case for any rectangle.

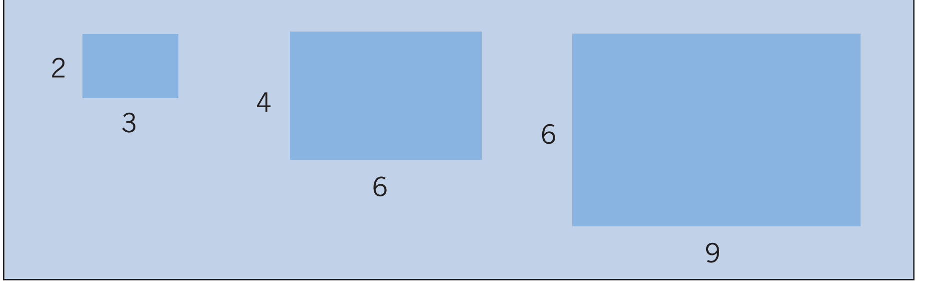
The teacher recognizes this as an opportunity to discuss proof. She asks her students the following questions:

- What do the second differences mean?
- Why are they constant?
- Will they always be constant?

This situation is ideal for giving students the opportunity to develop proofs. The students notice something interesting about their tables: Sara's conjecture hints at a relationship that the teacher knows will always hold. Moreover, the teacher wants the students to use this opportunity to understand more about the quadratic relationship between the rectangles' heights and areas. She asks the students to work in pairs and reminds them to focus on justifying their ideas.

When sharing their work, the students explain that the first differences represent how much area is added each time the rectangle grows. The second differences represent "the area of the amount added to the previous area," explains Bianca. Tai agrees: "It's the amount added to the amount added to the area." The students share drawings of growing rectangles and identify the additional area from one iteration to the next. For instance, when the original blue rectangle in **figure 3** grows to become

Fig. 1 A set of growing rectangles opens the door to a discussion about proof.



Edited by **Debra I. Johanning**, debra.johanning@utoledo.edu. This department provides information from research that teachers can use in their classrooms. It is intended to empower teachers with research-based information in their discussions with administrators, peers, and parents. Send submissions by accessing **mtms.msubmit.net**.

Fig. 2 A height and area table, with first and second differences, illustrates that the second differences are constant.

Height (cm)	Area (cm²)	First differences	Second differences
2	6		
		18	
4	24		12
		30	
6	54		12
		42	
8	96		12
		54	
10	150		

Fig. 3 A growing rectangle illustrates the second and third iterations.

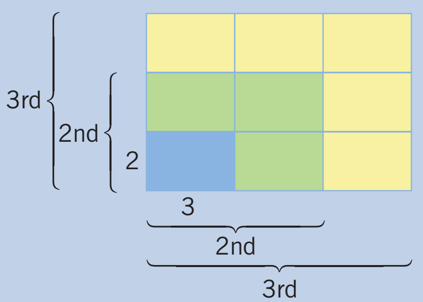


Fig. 4 These deductive arguments explain the identity 2(*n* – 1) = 2*n* – 2.

2(n-1) = 2n-2

This equation makes sense or true because when you multiply (2(n)-1) it's the same as 2n. You multiply -1 by 2 is equal to -2, which is the same as 2n(-2).

(a)

2(n-1) = 2n-2

For 2(n-1) is just like (n-1)+(n-1). If you put them togethess it's like 2n-2. You ~~ran~~ subtract 2 for the two ~~t~~ minus ones. Instead of n+n, you can do 2·n or 2n.

(b)

Source: Adapted from Boaler and Humphreys (2005)

the second rectangle, 3 additional rectangular units of area are added. When it grows again to become the third rectangle, 5 additional rectangles are added. The difference between the 3 additional rectangles and the 5 additional rectangles is 2 units of area, or the second difference, 12 square cm.

Pleased with how the students made sense of the first and second differences, the teacher then asks them to justify why the second differences remain constant for any set of rectangles whose dimensions grow proportionally. In this case, however, each student's proof relies on the use of a specific example.

Sara draws a 1 cm × 3 cm rectangle and shows that the second differences would be 6 square centimeters each time the height grows by 1 cm and the length grows by 3 cm. Other students make similar arguments with different rectangles, and some students rely on tables of values that they create, showing that the pattern holds for several different tables. The teacher asks the students whether they have any other way to form a proof, but none can answer. All the students are convinced that Sara's conjecture is true, but they cannot do anything beyond providing examples to prove it.

IS PROOF REASONABLE IN MIDDLE SCHOOL?

The scenario above may be familiar to teachers who try to help their students develop proofs. Forming deductive arguments that go beyond examples is difficult for students at all grade levels (Balacheff 1988; Knuth, Choppin, and Bieda 2009; Martin and Harel 1989). Students find examples-based arguments convincing, and they may also be convinced by arguments that appeal to authority, rely on perception, or otherwise depend on a justification that is not deductive (Harel and Sowder 2007). For instance, consider the following hypothetical proofs when students attempt to prove the identity $2(n - 1) = 2n - 2$ (a problem taken from Boaler and Humphreys 2005):

Janie: I tried it with 3, 5, 10, and 11 and it works every time, so it must be true.

Sam: I tried it with 1 through 5 and it worked every time. Then I tried it for a really big number, 1000, and for a really small number, 0, and it still worked. So it probably works for every number.

Malia: I know this is true because last year my teacher said that this is the distributive property, and you can just distribute the 2 across the parentheses.

Fig. 5 Two students justify their answers in different ways.

Question:
Say you have 5 test scores ranging from low to high. If you increase the highest test score, how will it affect the mean? Will it go up, down, stay the same, or does it depend on how much the score changes?

Andy's Response:
80, 82, 85, 90, 95 Mean = 86.4
80, 82, 85, 90, 100 Mean = 87.4

I tried it with three other examples, and in each case the mean went up. So increasing the larger score means that the mean will go up.

Hannah's Response:
Since we don't know what the scores are, I used variables:

$$\frac{a+b+c+d+e}{5} = \text{mean}$$

You know that they're all positive numbers because they are test scores. So say *e* is the highest score, then you're going to increase it by some amount, *f*. The new mean will be

$$\frac{a+b+c+d+e+f}{5},$$

which has to be bigger than it was before, because you're dividing a bigger number by 5. So it'll always go up.

Janie and Sam rely on examples to show that the identity is true; Malia relies on an authority-based argument. To help the class see the flaws in examples-based proofs, the teacher asks her students to consider how many examples they would need to prove the identity. Students reply with three, five, and less than ten. Travis, another student, argues differently:

> We thought that you have to try every single number there is because just because it works for, like, maybe twenty or thirty of them . . . it might not work for thirty-one, 'cause you haven't tried it yet. And so you have to find, to make sure that it works for everything, you have to try everything. So you have to find some way to explain it in words. (Boaler and Humphreys 2005)

The teacher highlights Travis's reasoning to help her students understand that examples would not be sufficient to prove the identity. This example leads to the creation of several viable proofs (see **fig. 4**).

Although proof can be difficult, middle school students and even younger can and do create mathematically appropriate proofs (Ellis 2007; Stylianides and Stylianides 2008; Zack 1997). In fact, proof has been shown to play an important role in promoting deep learning in mathematics (Hanna 2000; Yackel and Hanna 2003), and it can help support students' mathematical discoveries.

Consider Andy's and Hannah's attempts at proofs in **figure 5**. Andy's response is a typical examples-based argument, but Hannah produces a more general argument that does not rely on particular numbers. Another student asks her why the second formula has 6 variables but she continues to divide by 5, and Hannah explains:

> You can think of $e + f$ as just one number. It's just the same number but it has increased by f amount. So it's really still 5 test scores.

This led Isaac to realize the following:

> That means you don't even have to increase the largest value. If you increase any number, the mean has to go up, because it's like adding f to anything!

The teacher uses this discussion to launch a deeper exploration about which factors affect the mean and by how much.

Just as teachers' actions can help students move beyond examples-based reasoning and launch deeper investigations, the way that tasks are written can also encourage proof in middle school (Ball and Bass 2003; Bieda 2010). Students will be better poised to develop proofs if they encounter tasks that create the need for substantial mathematical reasoning.

TIPS FOR CLASSROOM PRACTICE

Middle school students need explicit support to develop appropriate proofs and move beyond arguments that rely on authority, perception, or examples. In addition to modifying tasks to provide more proof opportunities, there are other ways to support students' attempts as they begin to engage in proofs.

1. *Encourage students to make sense of existing justifications*

Students can learn a great deal about proving by making sense of others' proofs. In addition to encouraging them to share their reasoning, teachers can provide examples of both correct and incorrect justifications using a hypothetical student scenario. They

Fig. 6 This representation of $2n - 2$ may have come from a student's experience with algebra tiles.

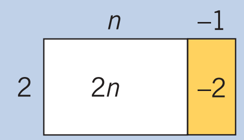
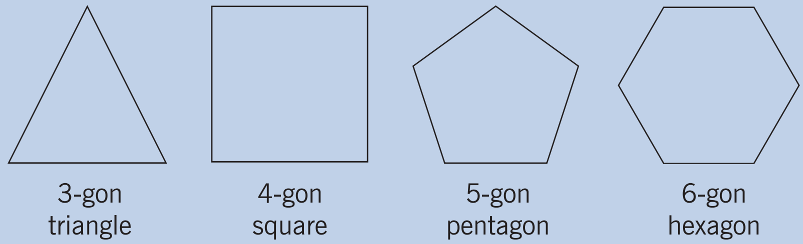


Fig. 7 A common textbook task can be extended to promote proof opportunities.

Here are some regular polygons. All the sides have the same length.



Complete the following table. What is the sum of the interior angles of a 36-gon?

Name	Triangle	Square			12-gon
Number of angles	3	4	5		
Measure of one interior angle	60°			135°	
Sum of all interior angles	180°		540°		1800°

Source: Adapted from deLange et al. (2006, p. 35)

can then introduce ideas in the guise of work from a student in a different class. When a justification comes from another student rather than from the teacher, students may be more willing to critique it and agree or disagree on the basis of the merits of the argument. For instance, for the identity $2(n - 1) = 2n - 2$, a teacher could introduce the proof that argues that $2(n - 1)$ can be expressed as $(n - 1) + (n - 1)$ (see **fig. 4**) and gauge students' reaction to this argument.

It can also, at times, be helpful to introduce an argument that is incorrect to assess what students find convincing. A teacher could introduce Sam's answer (see p. 523's dialogue) and ask whether his proof is more convincing than Janie's proof, because Sam uses extreme examples in addition to smaller numbers.

2. *Ask students to explain in a different way*

It is not uncommon for students to get stuck or be unable to produce a justification. Pushing students to move beyond rudimentary explanations can increase their capacities for proof (Ball and Bass 2003). Ask students to provide a different argument, to use a different representation, or to draw a picture. For the $2(n - 1) = 2n - 2$ identity, a teacher could ask students to draw a picture of $(n - 1)$ and $2(n - 1)$. This could lead to the image in **figure 4a.** Students who are familiar with algebra tiles and an area model of multiplication might make a picture like that in **figure 6**. At times, students may be too wedded to one particular representation to move to a more general argument, as in the scenario with the growing rectangles. In this case, a teacher might ask students to create a different picture that does not rely on a specific rectangle or to work with an algebraic representation.

3. *Point out what makes a justification valid*

In addition to understanding why particular proofs may not be sufficient, students need help understanding what constitutes a mathematically appropriate justification. Emphasize what makes a particular justification appropriate (Ball and Bass 2003). For instance, Hannah's answer (see **fig. 5**) is a powerful justification because it does not rely on specific numbers but instead makes a general argument. A teacher could emphasize the difference between Hannah's answer and another hypothetical student's answer, such as Andy's, and ask students to think about how they could further generalize Hannah's justification.

4. *Emphasize explaining "why"*

Showing that a conjecture is always true may not be strong motivation to create a proof, because students may already be convinced from trying several examples. Ask students to explain why a conjecture must be true; at that point, proof becomes a way for students to make sense of what is happening. For instance, the textbook example in **figure 7**, involving interior angles of polygons, asks students to develop a pattern but does not address why that pattern makes sense or where it comes from. Being able to explain why the formula $180°(n - 2)$ makes sense can encourage students to develop a proof that goes beyond showing that the pattern works with different examples. Extensions to promote student responses to the "why" question are shown in **figure 8**.

5. *Redirect students' attention to the contextual situation*

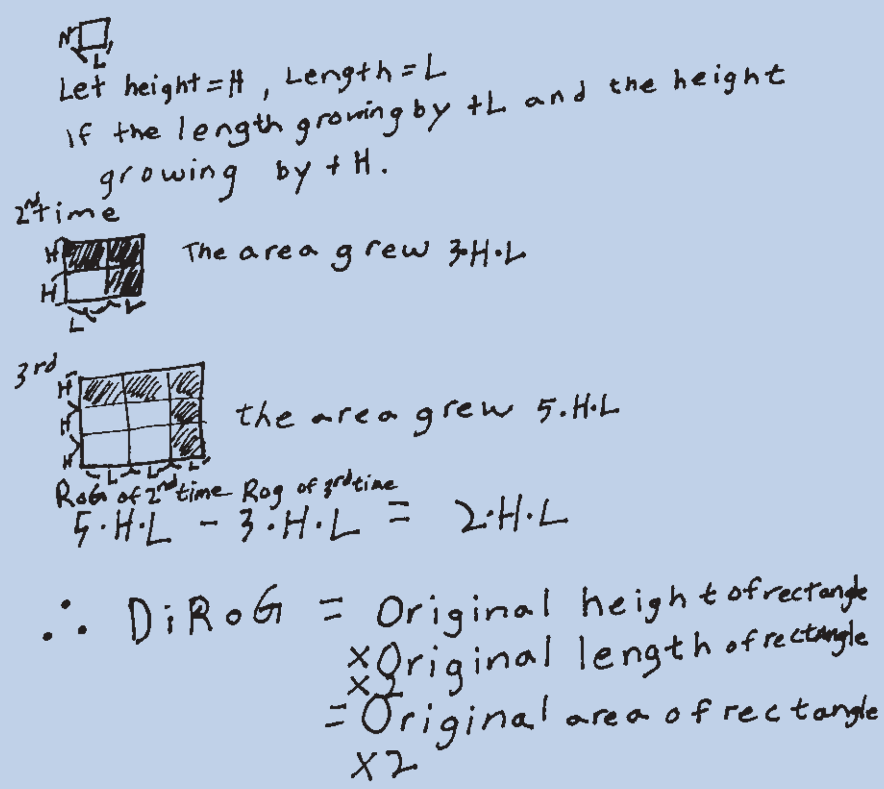
Once students extract a number pattern, it may be natural for them to focus on patterns rather than on the

Fig. 8 This extended task promotes generalization and proof.

Now that you know the sum of the angles of a triangle is 180°, how can you use this information to find the sum of the interior angles of a square?

1. Would your strategy work for a rectangle?
2. Would your strategy work for any quadrilateral?
3. Generalize your strategy to figures with a greater number of sides. How could you find the sum of the interior angles of a pentagon and a hexagon?
4. Provide a justification for why your strategy will always work.

Fig. 9 Once Daeshim establishes that the second differences for the area are constant, his explanation of the pattern he found becomes a proof.



situation that generated them. For instance, one group who works with the table in **figure 7** is able to develop a pattern and a general formula but cannot explain it. When their teacher redirects their attention to the figures, the students are able to formulate a justification.

The students who work with the growing rectangles reach a breakthrough when one student, Daeshim, is able to imagine a general $H \times L$ growing rectangle. He calls the first differences the "rate of growth" (or RoG) of the area; he calls the second differences the "difference in the rate of growth" (or DiRoG). He draws the picture in **figure 9** and determines that since he can calculate the difference in the rate of growth each time as $2HL$, it will not only always be constant but will also be twice the area of the original rectangle.

By pushing the students to develop an appropriate proof, even after they appear stuck, the teacher is able to foster a deeper understanding of the first and second differences for quadratic growth. The students are able to eventually create algebraic representations of the quadratic relationship between height and area.

Providing regular opportunities for students to prove and explicitly supporting students' emerging proof abilities will help middle school students not only become more adept at proving but also develop a deeper understanding of the mathematics they investigate.

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