Educational Studies in Mathematics (2020) 104:87–103
https://doi.org/10.1007/s10649-020-09951-6

Scaling-continuous variation: supporting students' algebraic reasoning

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Published online: 18 May 2020

Abstract
This paper introduces a new mode of variational and covariational reasoning, which we call scaling-continuous reasoning. Scaling-continuous reasoning entails (a) imagining a variable taking on all values on the continuum at any scale, (b) understanding that there is no scale at which the continuum becomes discrete, and (c) re-scaling to any arbitrarily small increment for x and coordinating that scaling with associated values for y. Based on the analysis of a 15-h teaching experiment with two 12-year-old pre-algebra students, we present evidence of scaling-continuous reasoning and identify two implications for students' understanding of rates of change: seeing constant rate as an equivalence class of ratios, and viewing instantaneous rate of change as a potential rate. We argue that scaling-continuous reasoning can support a robust understanding of function and rates of change.

Keywords Student reasoning · Algebra · Middle school · Teaching experiments

1 Introduction: the importance of variation for function understanding

Functions and relations comprise a critical aspect of algebra, with recommendations for supporting students' algebraic reasoning advocating the introduction of functional relationships in the middle grades (e.g., National Governor's Association Center for Best Practices, 2010; U.K. Department for Education, 2009). Despite the importance of functional reasoning, however, research indicates that students exit secondary school viewing functions in terms of symbolic manipulations rather than as a model of dynamic situations (Stephens, Ellis, Blanton, & Brizuela, 2017). These findings highlight the need to better support students' emerging function concepts, particularly in terms of understanding functions as representations of

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variation. Currently, typical curricular and pedagogical approaches rely on the correspondence view, which treats a function as a fixed relationship between the members of two sets. This static treatment is widespread in secondary instruction; for instance, Thompson and Carlson (2017) reviewed 17 US secondary textbooks, ranging from Algebra I through Precalculus, and found that all relied on a correspondence definition of function.

A contrasting approach to supporting students' functional thinking is an emphasis on variational and covariational reasoning (Thompson & Carlson, 2017). Researchers argue that attending to and coordinating changes in quantities that continuously covary is critical for students' development of a robust understanding of function, constant and varying rates of change, and the foundational ideas of calculus (e.g., Carlson, Smith, & Persson, 2003). Further, situating functional exploration within contexts leveraging covarying quantities that enable visualization, manipulation, and prediction has been found to foster students' abilities to reason flexibly about dynamically changing events (Castillo-Garsow, Johnson, & Moore, 2013). Covariation is an examination of coordinated changes between x- and y-values (Confrey & Smith, 1995; Saldanha & Thompson, 1998). Confrey and Smith (1995), for instance, defined a covariation approach as moving operationally from y_{m} to $y_{\mathrm{m+1}}$ in coordination with movement from x_{m} to $x_{\mathrm{m+1}}$. Saldanha and Thompson (1998) then extended this idea to characterize covariation as imagining two quantities changing together, which, in turn, is dependent on the ability to envision each quantity varying. From this perspective, covariation is the coupling of two quantities, which enables tracking either quantity's value with an explicit understanding that at every instance, the associated quantity has a corresponding value (Saldanha & Thompson, 1998).

In this paper, we introduce a new mode of variational and covariational reasoning, which we call *scaling-continuous covariation*. Scaling-continuous covariation entails imagining the continuum as infinitely zoomable, coupled with the understanding that one can re-scale to any arbitrarily small increment for x and coordinate that scaling with associated values for y. This paper reports on a study addressing the following research questions: (a) How does scaling-continuous reasoning differ from prior examples of covariational reasoning? (b) What are the implications of scaling-continuous reasoning for students' understanding of functions and rates of change? In the findings below, we argue that scaling-continuous reasoning can support productive ways of thinking about key function ideas, including constant and instantaneous rates of change.

2 Theoretical framework and relevant literature: rate reasoning and covariational reasoning

Stroup (2002) introduced the term qualitative calculus, which entails an informal introduction of calculus concepts, such as rates of change, to students in upper elementary and middle school. Given that calculus is a dynamic discipline (Tall, 2009), an approach emphasizing rate reasoning can be fruitful for supporting students' emerging function understanding (Carlson et al., 2003). This can be especially important given the prevalence in many European countries of introducing calculus in secondary school (Artigue, 2002; Maschietto, 2004). Rate of change, however, remains a difficult concept for both secondary students (e.g., Herbert & Pierce, 2012) and university students (e.g., Ubuz, 2007). There is some evidence that situating students' early exposure to function and rate reasoning in varying (rather than constant) rate contexts can be beneficial (e.g., Herbert & Pierce, 2012; Stroup, 2002). Further, emphasizing covarying quantities in rate situations can support students' understanding of rate as a relationship, rather than as the outcome of a calculation (Herbert & Pierce, 2012). Consequently, we chose to introduce dynamic contexts in which students could explore constant and varying rates of change in order not only to foster robust rate conceptions, but also to support an understanding of continuous function concepts.

Representing continuous relationships is challenging for students, and they often resort to discrete graphs to depict continuous phenomena (de Beer, Gravemeijer, & van Eijck, 2015). It is also not uncommon to approach continuous functions additively by comparing the changes in the output variable with respect to equal increments of the input variable (Kertil, Erbas, & Cetinkaya, 2019). However, there have been reports of successes in students moving from discrete to continuous representations (Yerushalmy & Swidan, 2012). For instance, de Beer et al. (2015) found that 5th-grade students could bootstrap their discrete reasoning about speed to make sense of continuous graphs and, ultimately, reason qualitatively about instantaneous speed. In order to make sense of students' rate reasoning for continuous functions, we leveraged ideas from Castillo-Garsow's (2012) and Thompson and Carlson's (2017) covariational reasoning frameworks, which we detail below.

2.1 Covariational reasoning frameworks

Thompson and Carlson (2017) provided an overview of research on student reasoning with continuous functions that synthesizes the work of a variety of researchers over the past several decades (e.g., Carlson et al., 2003; Castillo-Garsow, 2012; Castillo-Garsow et al., 2013; Saldanha & Thompson, 1998; Thompson & Carlson, 2017). Our work here is situated in the context of this body of research and builds upon these frameworks.

Chunky-continuous variation and covariation Castillo-Garsow (2012) distinguished between two different images of change as students reason about variation: chunky and smooth reasoning. According to Castillo-Garsow et al. (2013), a *chunky* image of variation has two distinguishing features: "A unit chunk whose repetition makes up the variation, and a lack of image of variation within the unit chunk" (p. 33). One generates change by sequencing equal-sized chunks and measures change by counting the number of elapsed chunks. Castillo-Garsow and colleagues emphasized that a key aspect of chunky thinking is that one thinks *in* intervals, but not *about* intervals; intermediate values within a chunk may exist, but do not receive explicit attention.

Thompson and Carlson (2017) called this *chunky-continuous* variation: change occurs in completed chunks, but there is no clear image of how variation occurs within the chunk. For chunky-continuous covariation, the student imagines corresponding chunks in the other covarying quantity as well. When using chunky-continuous reasoning, a student may be able to reason with different chunk sizes; a given chunk is not necessarily indivisible. Rather, one chooses a chunk size and measures change in units of that chunk.

Smooth-continuous variation and covariation *Smooth* variation relies on an image of change experientially as it occurs (Castillo-Garsow et al., 2013); it is imagined in the present tense. In contrast, chunky thinking is in the past tense, imagining that the change has already happened and is now being analyzed. These authors add, "Smooth images of change are not the same as chunky images of change cut up really small. Smooth images of change involve an entirely different conceptualization of variation" (p. 34). The difference between smooth and chunky reasoning is like the difference between the experience of watching a movie and that of the movie-maker editing a sequence of already-filmed frames.

Thompson and Carlson (2017) described *smooth-continuous* variation in terms of projecting an image of one's own experiential time to a time period within the mathematical context. One imagines a value varying as its magnitude increases in bits while simultaneously anticipating smooth variation within each bit, passing through all of the intermediate values within any given bit. Smooth-continuous covariation then involves smooth variation in both quantities simultaneously. Furthermore, one can choose to consider change by intervals "while anticipating that within each interval the variable's value varies smoothly and continuously" (2017, p. 430). Thompson and Carlson therefore consider smooth-continuous reasoning to be more powerful than chunky-continuous reasoning, in that a student who can use smooth-continuous reasoning can also reason about change in chunks if needed, while a student who can use chunky-continuous reasoning may not be able to reason smoothly. Castillo-Garsow et al. (2013) make no such claim, and they treat chunky thinking and smooth thinking as different types of imagery between which a student might switch back and forth.

Both accounts of smooth reasoning agree that it requires reasoning in terms of motion, always entailing imagining "something moving" (Thompson & Carlson, 2017, p. 430). A varying quantity is imagined as being tacitly parameterized by conceptual time. Thompson and Carlson (2017) note that this is the same imagery Isaac Newton appealed to when he defined a variable quantity to be a "fluent" that depended on and changed with time.

2.2 Scaling-continuous variation and covariation

We now define a new image of change: *scaling-continuous variational and covariational reasoning*. The fundamental image involved in this type of reasoning is zooming or scaling. Scaling-continuous variation entails imagining that a variable takes on all values in any continuum of values, and assuming this will always be the case no matter how much you zoom in. The image is that the continuum is infinitely "zoomable," in that the process of zooming never reveals discrete atoms, holes, or values that the variable skips. The continuum is imagined as always being a continuum at any scale.

Scaling-continuous covariational reasoning imagines scaling to see any arbitrarily small continuum of values for one variable, and that this increment will always correspond to a continuum of associated values for the other variable. For instance, one can imagine a window of x-values growing, and the corresponding window of y-values simultaneously growing as a correspondence between increments of x and y.

Scaling-continuous reasoning does not require an image of motion or assume an underlying time parameter. In this way, it differs from smooth reasoning and is similar to chunky reasoning. Like chunky reasoning, it also treats change as having already occurred; reasoning about change is not grounded in the reasoner's experiential time. Yet scaling-continuous reasoning is unlike chunky reasoning in several ways. First, its fundamental image is of zooming, not of traversing an interval of change using a chosen chunk. Second, although a person using chunky reasoning does not attend to variation *within* a chunk, a person using scaling reasoning assumes there is always a correspondence between the values of the two variables at every scale within every chunk.

The third difference is the different form of generalizing each type of reasoning affords. Chunky reasoning entails imagining movement across a domain in chunks, so it can afford a generalization of some feature of the covariation across chunks. This could allow a student to imagine that feature being present in every chunk of the situation's domain. On the other hand, scaling-continuous covariational reasoning employs the image of repeated zooming or rescaling, which could enable one to generalize a feature of the covariation across *all* scales, including, potentially, the infinitesimal scale.

To illustrate this idea of generalizing across scale, we appeal to G. W. Leibniz' accounts of covariation in calculus. There are some notable parallels between scaling-continuous reasoning and the imagery G. W. Leibniz used when writing about infinitesimal calculus. We do not wish to speculate about what Leibniz thought, but rather to use Leibniz' imagery to motivate the idea of how scaling-continuous reasoning affords generalizing across scale.

Just as Newton's imagery of fluents has parallels with smooth reasoning, Leibniz' imagery of covariation has parallels with scaling-continuous reasoning. Rather than treating two covarying quantities as flowing simultaneously, Leibniz typically attended to the correspondence between increments of the two quantities, particularly infinitesimal increments (Bos, 1974). He distinguished among types of these increments based on their relative *scales*. For instance, for differential and integral calculus, Leibniz used infinitesimal increments. Starting with an algebraic relationship between the values of two variable quantities x and y, his differential calculus was then a way to derive a new equation describing the relationship between infinitesimal increments of the two quantities, dx and dy.

This idea can be illustrated by Leibniz' summary of the product rule:

> $d(xy)$ is the same as the difference between two adjacent xy, of which let one be xy, the other $(x + dx)(y + dy)$. Then, $d(xy) = (x + dx)(y + dy) - xy$, or $xdy + ydx + dxdy$, and this will be equal to $xdy + ydx$ if the quantity $dxdy$ is omitted, which is infinitely small with respect to the remaining quantities, because dx and dy are supposedly infinitely small (namely if the term of the sequence represents lines, increasing or decreasing continually by minima). (From Leibniz' *Elementa*, quoted in Bos, 1974, p. 16.)

First, we note the image of correspondence between the bits of two covarying quantities. Leibniz described a new variable quantity, xy, and he then sought to derive an equation describing the correspondence between an infinitesimal bit of this quantity, $d(xy)$, and infinitesimal bits (dx and dy) of the other two quantities x and y. Nothing about these increments is moving or changing, although their values depended upon where on the curve they were taken. Although Leibniz did not appeal to motion, the idea of covariation here entails the notion that every increment of one quantity, no matter how small, corresponds to an increment of another covarying quantity.

The second image we note is that of scaling. In the above example, Leibniz dismissed the quantity $dxdy$ because it is infinitely small even in comparison with other infinitely small quantities such as dx and xdy. Leibniz developed a scheme of orders of the infinitesimal and the infinite in order to systematize this idea of scaling. At the finite scale, infinitesimals such as dy are negligible, but at the first-order infinitesimal scale, they become significant, with second-order differences still negligible. Imagining correspondence at these different scales was crucial to a coherent system of calculus for Leibniz. At each such scale, the continuum was still continuous (Bos, 1974). The image of scaling can afford infinitesimal increments by generalizing features from finite increments in order to envision infinitesimal increments. One could imagine a process of infinite zooming, with the reification of such a process supporting an image of an increment at the infinitesimal scale that inherits properties from the finite cases (Ely, 2011).

One of the properties that can be projected onto infinitesimal increments is that of local straightness. For instance, Leibniz spoke of a curve as a polygon with infinitesimal sides, so that a difference triangle can be imagined at the infinitesimal scale with a straight hypotenuse of slope dy/dx. Scaling-continuous reasoning could afford, but by no means compels, this generalization of straightness to the infinitesimal scale. Differentiable curves appear straighter and straighter as you zoom in on them more and more. By attending to this property as one generalizes across scale, one might project onto the infinitesimal scale the image of local straightness. This is precisely the type of generalization Tall (1997) and Maschietto (2004) have sought to promote in their approaches to calculus through local linearity and the global/local game, respectively. Tall (2009), for instance, relied on the notion that a differentiable graph under infinite magnification is a straight line. If one zooms in dynamically on a graph with very small values of dx and dy, then the magnified graph looks like a straight line so that the graph and its tangent become indistinguishable. Both Tall (2009) and Maschietto (2004) leveraged the notions of embodiment and zooming to foster a shift from a global to a local perspective. These approaches, therefore, employ scaling-continuous imagery, which could afford the image of infinitesimal increments.

As illustrated in the "Results" section, this image of local straightness is not the only image that can be generalized across scale. If a student attends to different elements of the situation of covariation, then scaling-continuous reasoning might instead afford the projection of the image that a curve is always curved, even at infinitesimal scales. Scaling-continuous reasoning affords generalizing across scale, but does not by itself dictate what properties are generalized.

3 Methods

The study reported in this paper was part of a larger 3-year project aimed at understanding students' generalization processes in algebra, advanced algebra, and combinatorics. As part of this project, we implemented an exploratory teaching experiment in order to investigate students' generalizations of linear, quadratic, and higher-order polynomial functions from a rate-of-change perspective. The results of this study then supported the design of an instructional sequence for a series of larger-scale design experiments on function. The findings in this paper are from the first teaching experiment.

3.1 Participants and the teaching experiment

We conducted a 10-day, 15-h videoed teaching experiment (Steffe & Thompson, 2000) with two 12-year-old students in general mathematics (neither had yet taken algebra). We assigned gender-preserving pseudonyms to each student, Wesley and Olivia. The first author was the teacher-researcher, and a project member observed each session, which lasted between 1 and 2 h. The project team met daily to debrief.

We developed tasks to support a conception of linear growth as a representation of a constant rate of change and quadratic growth as a representation of a constantly changing rate of change. The tasks emphasized these ideas within the contexts of speed and area. The area tasks presented "growing rectangles," "growing stair steps," and "growing triangles" via dynamic geometry software, in which the students could manipulate a figure by extending the length and observing the associated growth in area (Fig. 1). Table 1 provides the mathematical topics and contexts addressed each day during the teaching experiment.

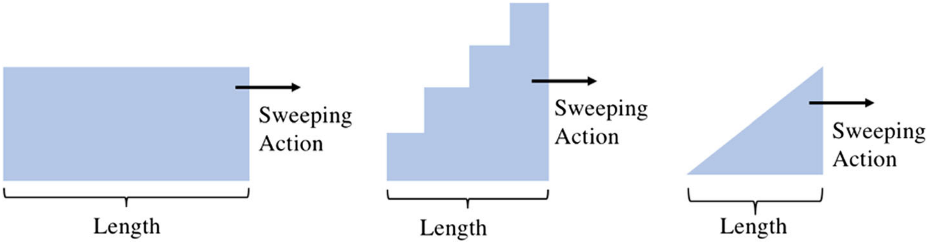


Fig. 1 Growing rectangle, stair step, and triangle tasks

Table 1 Overview of the teaching experiment unit

Day	Mathematical topics	Contexts
1	Linear growth, average rate of change	Speed
2	Linear growth, average rate of change	Speed, growing rectangles
3	Linear and piecewise linear growth	Growing rectangles, stair steps
4	Quadratic growth, average rate of change	Growing triangles
5	Quadratic growth, identifying constantly changing rates of change	Growing triangles
6	Quadratic growth, instantaneous rates of change	Growing triangles, trapezoids
7	Instantaneous rates of change	Growing rectangle, triangles, trapezoids
8	Cubic growth	Growing cubes and rectangular prisms
9	Cubic growth	Growing cubes and rectangular prisms
10	Higher-order polynomial functions	Growing 4D and n-dimensional figures

The researchers' goals informed the design and sequencing of the tasks. One goal was to create opportunities for students to reason about rates of change in progressively more sophisticated ways. The growing rectangle task provided an opportunity to establish that as the length and area grow together, the rate of change is fixed regardless of increment size. The stair step task had a fixed rate of change within each stair but increased by the same amount from stair to stair. The intent was to help the students begin to form the language and tools to model situations with non-constant rates of change.

The growing triangle task models quadratic growth. In this case, there are no periods of constant rate of change that can be calculated by dividing the displacement in area by the displacement in length. One can only calculate the average rate of change on that interval. We anticipated that students would partition the triangle into vertical columns of equal increments and reason about how the rates of change in area increased from one interval to the next. After working with different increments and varying their width, we hoped to encourage the students to begin reasoning informally about instantaneous rates of change by anticipating a rudimentary limiting process. Scaling-continuous reasoning emerged as a way of viewing functions that was adapted in part to these instructional materials and objectives.

3.2 Analysis

We employed retrospective analysis (Steffe & Thompson, 2000) in order to characterize the students' conceptions throughout the teaching experiment. We transcribed teaching session and then produced a set of enhanced transcripts that included all verbal utterances, images of students' work, descriptions of relevant gestures, and other non-verbal actions. Relying on the constant comparative method (Strauss & Corbin, 1990), we then analyzed the data in order to identify (a) students' forms of covariational reasoning and (b) students' conceptions of constant and changing rates of change. For the first round of analysis, we drew on Thompson and Carlson's (2017) framework of variational and covariational reasoning. We coded to infer categories of variation/covariation based on students' talk, figures, gestures, and task responses. We also developed emergent codes for students' understandings of constant, changing, and instantaneous rates of change. The first round then guided subsequent rounds of analysis in which the project team met to refine and adjust the codes in relation to one another. This iterative process continued until no new codes emerged. The final round of analysis was descriptive and supported the development of an emergent set of relationships between the students' covariational reasoning and their conceptions of constant, changing, and instantaneous rates of change.

4 Results

We focus on the distinction between two forms of reasoning in the teaching experiment, chunky-continuous and scaling-continuous covariation. Although these were not the only forms of reasoning we observed, they were the most prevalent forms that persisted throughout the teaching experiment. Further, we highlight chunky-continuous and scaling-continuous reasoning as a way to address the unique characteristics and affordances of scaling-continuous covariation as distinct from chunky-continuous covariation. In the sections below, we introduce evidence of the two forms of reasoning and then discuss the ways in which scaling-continuous covariation afforded generalizations about constant and instantaneous rates of change.

4.1 Chunky-continuous reasoning

On the fourth day of the teaching experiment, the students watched a video of a triangular region that grew in a smooth, continuous motion from left to right (Fig. 2a). The teacher-researcher (TR) asked the students to construct a graph to show the relationship between the total accumulated area and the length swept. Wesley explained his graph (Fig. 2b) by discussing segments of inches: "It's curved because as the length keeps going, for every inch it covers more area." Chunky-continuous covariation entails imagining each quantity's value changing by intervals of a fixed size. A student reasoning in this manner could re-chunk to different sizes, but would still lack an image of variation within a given chunk. In this case, Wesley considered amounts of length in 1-in. chunks, and he anticipated an amount of growth in area associated with each chunk. However, Wesley did not show evidence of considering how the length and area values accumulated together *within* each chunk.

In explaining the second graph Wesley produced on graph paper (Fig. 2c), he noted, "every inch it goes, it, like, it goes, it covers more area for that inch so it keeps getting steeper." The teacher-researcher asked Wesley whether the segments connecting the points were straight or curved:

W: I think they'd be a straight line too.
TR: Okay. When you say every inch it goes, it covers more area, is that true for only inches or is it true for any sort of increase?
W: Any sort of increase I think.

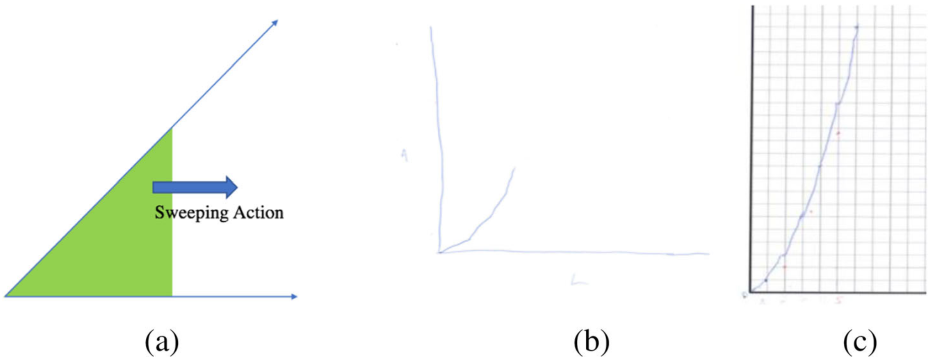


Fig. 2 A static image of the triangle's area swept from left to right (**a**) and Wesley's graphs (**b**, **c**)

Wesley stated that his image of growth would hold for "any sort of increase," suggesting that he may have been able to understand that the growth phenomenon was not dependent on the particular intervals he chose. The straight-line segments, however, suggest that any values occurring within chunks were tacit. A student reasoning with smooth-continuous covariation might instead create a curved graph that would reflect an image of continuous coordinated changes in x and y, understanding that for any arbitrary increment, both quantities will covary with smooth variation, moving through all values within the increment.

Olivia also showed evidence of chunky-continuous reasoning. When the teacher-researcher presented a growing triangle with a height-to-length ratio of 2:5, she invited the students to graph its area versus length. Olivia produced a piecewise linear graph with 1-in. increments for length, stating, "each line between each increment is just getting steeper":

> O: If you made the increments even smaller, like into 0.1 as your first point, then I think it'd be, all the little lines together I think they'd make a very subtle curve, but relatively straight. So when I did it with the increments as 1, I see them as straight, *but if they were smaller they might look as if they were curved to make one big curve.*

Olivia reasoned from one increment to the next, noting changes in steepness per line segment. She also affirmed that using smaller increments would change the appearance of the graph, an important hallmark of chunky-continuous covariation. When engaged in chunky reasoning, a student can "re-chunk" to different increments, but those increments still have a measurable length. Olivia reasoned that each interval was represented by a line segment; she could adjust the size of her increments, but without altering her view of the nature of change *within* each increment.

4.2 Scaling-continuous reasoning

On day 5 the teacher-researcher showed the students a video of another growing triangle, but this time asked them to draw a sketch of the area-length graph simultaneously as the video played. Wesley and Olivia worked together and both produced a smooth curve (Wesley's is shown in Fig. 3). The teacher-researcher asked Wesley why the graph was curved, in contrast to his prior piecewise graphs:

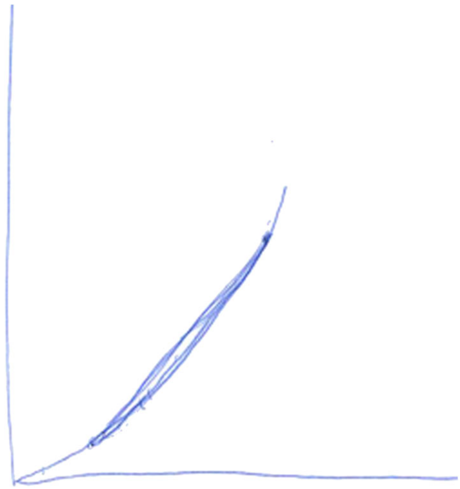


Fig. 3 Wesley's smooth area-length sketch reproduced from a dynamic sketch

W: Because, like, we were doing like big increments like here to like here (marks two points on the curve) and if you kind of draw a straight line (draws a line between the points) it's like not exactly on the curve. But if you add the tiny increments, like in-between, then it curves out.

The teacher-researcher then asked Wesley what the graph would look like between two points that were "super close together": would it be curved or straight? Wesley indicated that it would be curved, explaining that "there's tiny points *in between* those tiny points." The teacher-researcher further asked what would happen between two infinitesimally close points:

TR: What if I picked two points that were so close together that I couldn't, you couldn't even see the difference? They were just so close together there's like an infinitesimal difference in between them. Would the connection between them be a straight line, or a curve still?
O: Like the tiniest ones?
TR: Uh huh.
O: Then it would be a straight line.
TR: (Turns to Wesley). What do you think?
W: I think it'd be more of a curve because I think like it goes on infinitely, kind of, the points. So if you zoomed in really close on those it would like look like that and then *in between those there's still more points and it goes on forever*.
TR: (Turns to Olivia). What do you think?
O: I still think it'd be a straight line because to me it's just a whole bunch of little straight lines and so like to me *it would eventually stop* because you're graphing the triangle's, like, placing.

One way to distinguish Olivia's position from Wesley's is through the lens of potential versus actual infinity, a distinction that originates in Aristotle's work. Potential infinity is characterized by an ongoing process repeated over and over without end (Núñez, 2003). Núñez characterized this process by describing the action of imagining an unending sequence of regular polygons with more and more sides. The process, at any given stage, encompasses only a finite number of repetitions. As a whole, however, it does not end and therefore lacks a final resultant state. In contrast, actual infinity characterizes the infinite process as a realized thing. Even though the process lacks an end, it is conceived as being completed and as having a final resultant state. Following the same example, one can imagine an end *at infinity* "where the entire infinite sequence *does have* a final resultant state, namely a circle that is conceived as a regular polygon with an infinite number of sides" (p. 52, emphasis original). Similarly, one could imagine a curved graph as having an infinite number of straight sides (as Leibniz did), but, as we explain next, we do not believe that Olivia used actual infinity when imagining a curve made up of "a whole bunch of little straight lines."

Olivia could re-imagine the individual chunks to be smaller and smaller, but that process did not have a realized end state whereby the smooth curve and the piecewise linear approximation were one and the same. Olivia's "bunch of little straight lines" were not necessarily *infinitely* many straight lines; hence, Olivia noted that "it would eventually stop." In contrast, Wesley's claim that the segment would be curved is consistent with a notion of actual infinity. He spoke of the ability to zoom in such a way that it "goes on forever." Wesley could imagine points in between points, at any scale, even zooming in indefinitely. Wesley consequently treated the quantities' values as varying continuously, taking on all possible values within the interval, even if the interval was infinitesimal. An always-curved curve is the generalization Wesley made by employing actual infinity; he projected the property of "curving over every interval" onto infinitesimal intervals. In contrast, one could instead generalize, as did Leibniz, that the curve is made up of infinitely many straight lines.

The difference between Olivia and Wesley's reasoning illustrates an important distinction between scaling-continuous and chunky-continuous covariation. We also note that Wesley did not use smooth-continuous covariation, because his language contained no references to motion. His imagery was of scaling, not of a variable moving and tracing out values as it moved.

4.3 Affordances of scaling-continuous reasoning

In the following sections, we outline a critical way in which scaling-continuous covariation supported Wesley's thinking about rates of change: Namely, he was able to construct a multiplicative rate object by appealing to a figure's height, for both constant and changing rate of change figures. Below, we illustrate how this occurred in relation to Wesley's scaling-continuous covariational reasoning, and contrast his thinking with what is afforded by chunky-continuous covariation.

Constant rates of change Both students could reason about length and area growing together, but their conceptions of the ratio of area to length differed. Olivia conceived of this relationship as a ratio, by considering an amount of elapsed area in comparison with an amount of elapsed length. Wesley, in contrast, developed an understanding of the ratio as a rate of change. For instance, on day 6, Olivia and Wesley examined a dynamic figure in which the area was swept out under the curve in Fig. 4. The teacher-researcher positioned the mouse near the left end, where the figure's height is 3 cm. She asked the students, "At what rate is the area increasing in this portion?"

Olivia answered, "The area, if you break it into parts, the area would just keep growing." She concluded that the rate would be "consistent" because the amount of area gained for each

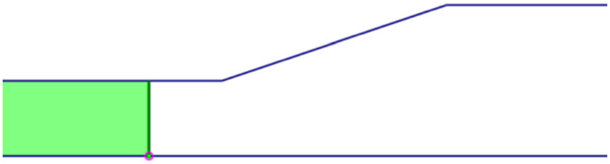


Fig. 4 Dynamically growing figure depicting constant and constantly changing rates of change

"part" was the same: Olivia compared the accumulated area across same-length increments. In contrast, Wesley answered 3 cm^2 per cm swept. He explicitly referenced both quantities, and when justifying his answer, Wesley said, "Because the height is 3 centimeters." As we detail below, we believe that Wesley's appeal to the height offers evidence that he conceived of the ratio of 3 cm^2 of area for 1 cm of length to be a rate.

Thompson and Thompson (1992) described a rate as a reflectively abstracted constant ratio. A ratio is a multiplicative comparison of two quantities; a rate requires viewing two such quantities as changing together, and treating the collection of equal ratios they generate as a single quantity of its own. It symbolizes the ratio structure as a whole while giving prominence to the constancy of the result of the multiplicative comparison. Wesley's appeal to the height, a single quantity, to justify the rate suggests that he saw it as a representation of the collection of equal ratios. In order to understand the height as a rate, Wesley needed to have an image of change such that 3 cm^2:1 cm represented an equivalence class of ratios.

Wesley appealed to the height as a representation of the area's rate on a number of other tasks. For instance, later on the same day, the teacher-researcher adjusted Fig. 4, making the rightmost region's height 10 cm. Wesley said the area's rate of change there was 10 cm^2 per cm, because its height was 10 cm. On yet another similar task where the rectangular region's height was 4 cm, Wesley justified the area's growth rate to be 4 cm^2 per cm because its height was always 4 cm regardless of how much length had been swept out. Wesley generalized that all of the ratios were instantiated in the same rectangle height, which did not depend on a specified amount of length. This idea later supported his reasoning with instantaneous rate of change, as we describe in the next section.

Wesley's work on the 4-cm rectangle task also provided additional evidence that he had constructed a rate. The teacher-researcher asked the students to create multiple ratios representing a 4-cm^2 per cm rate of change. Both students produced a number of equivalent ratios. Olivia, however, could not produce ratios for length increments less than 1 cm, and she did not perceive 8 cm^2:2 cm to be the same as 4 cm^2:1 cm. Olivia struggled to disentangle the rate of growth of the area with the amount of length lapsed. Wesley generated ratios that included length increments greater than and less than 1 cm, such as 12 cm^2:3 cm, 8 cm^2:2 cm, 2 cm^2:0.5 cm, and 0.4 cm^2:0.1 cm. More importantly, Wesley indicated that he would be able to create an equivalent ratio given *any* elapsed length, even an unspecified length of x, simply by multiplying it by 4. In contrast, in order to determine the area for an unspecified length smaller than 1 cm, Olivia needed to first find an elapsed length as some fraction of 1 cm, and then take the height of 4 cm and multiply it by that same fraction to get the proportional amount of area. This method depended on identifying an extant length first. Wesley's ability to appeal to only the height of the rectangle, rather than an increment representing an elapsed amount of length, suggests that he was able to see the area-to-length ratio as a rate.

It is plausible that Wesley's reasoning about constant rate was a consequence of a generalization enabled by his scaling-continuous covariational reasoning. He coordinated increments of length with corresponding increments of area across smaller and smaller scales of length. The concept of actual infinity could also assist with the conception of a height as a rate. If one can imagine smaller and smaller length scales and then characterize that process as having a resultant state, the resultant state would be not a column with a tiny amount of length, but a line (i.e., a height) with no associated length. Wesley could then generalize across all scales the property that the ratio was always constant.

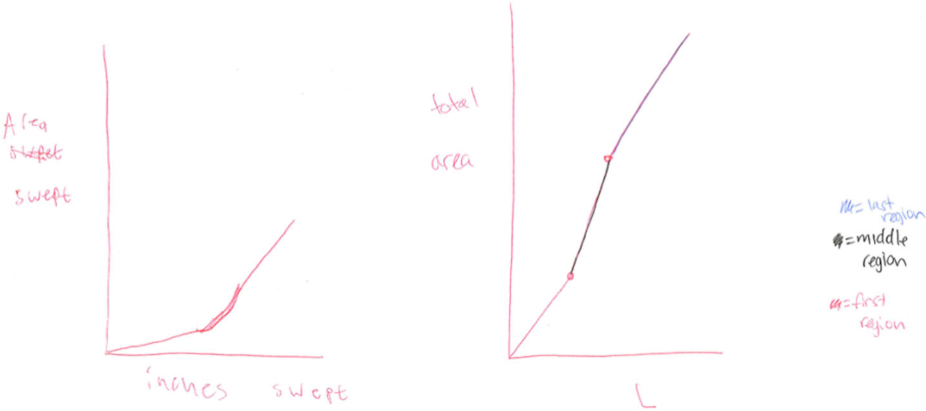
Instantaneous rates of change Wesley began to identify an instantaneous rate as also determined by a figure's height at the relevant location. In an initial activity to investigate the instantaneous rate of change, the teacher-researcher directed the students' attention to the trapezoidal middle region in Fig. 4. She asked the students to describe the rate of change of the area at the halfway mark of this region (where the height was 4.5 cm). Olivia explained that she saw the rate as "constantly getting larger than the previous increment," indicating a need to compare an amount of area for an elapsed increment with that for a previous elapsed increment. The teacher-researcher pushed Olivia to be more specific by asking, "Could we come up with a rate?" Olivia struggled to make sense of this question. She asked, "From, throughout the time?" and the teacher-researcher replied, "No, at that moment." Olivia answered, "It's not increasing, because you're not really going."

Wesley disagreed with Olivia's answer, stating, "I think it might be zero point – or, one-third plus 3 centimeters. *Because the slope of this is one-third*, but then you also have to take into account the rectangle." His answer reflected the fact that he viewed the trapezoidal region as a triangle atop a rectangle. The triangle's slope was 1/3, which means its area grew at a changing rate, according to 1/3 of its length. Thus, the area's rate of change halfway through the region was the rectangle area's growth rate, 3 cm^2 per cm, plus 1/3 of 4.5 (cm^2 per cm).

The teacher-researcher then asked for the instantaneous rate of change one-third of the way through the trapezoidal region. Wesley immediately said 4, which was the height of the trapezoid at that location: "Because it'd be 3 plus 1, since if it's a third [the slope] you can divide it 3 into 3 parts, which is 1." Likewise he said the area's instantaneous growth rate two-thirds of the way through the trapezoidal region the rate was 5 cm^2 per cm, again because the height of the figure was 5 cm there. In each case, Wesley used the slope of the region to determine the figure's height for a given swept length, and then said the area's rate of change was the region's height.

Wesley had initially generalized that the area's rate was the figure's height for rectangles, which have constant rates of change. One way to develop the more general understanding for a non-rectangular region is to consider how much area would be produced if the length were to begin sweeping out a bit. One could think of a bit as 1 cm, as Olivia did, but it is not necessary to sweep an entire centimeter in order to calculate the area's rate of change. Imagining this rate occurring at any scale, even an infinitesimal one, could allow one to see the rate as a "height," not as an extensive quantity, but as a ratio. Once the length sweeps out any amount, it turns the potential rate into an amount of area depending on how much length has been swept. All that matters is the region's height just at the moment when the area begins to grow.

A comparison of the students' graphs suggests that Wesley's image was of the area's growth rate continuously increasing throughout the sloped region, as evidenced by his attempt to draw a curved middle portion (Fig. 5a). In contrast, Olivia's graph represented the area's growth rate as constant (Fig. 5b). Olivia's explanation of the trapezoidal region on her graph suggests that she relied on chunky-continuous covariation: "In between each very small increment it's still growing, but you have to connect the two increments together *because it's the same shape*. It keeps growing at the same rate relatively." This suggests that Olivia did



(a) (b)

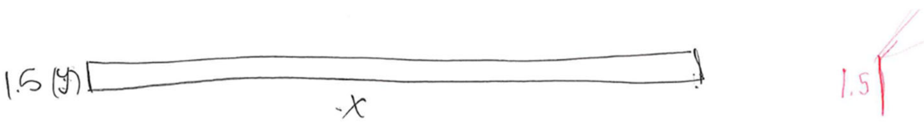
Fig. 5 Wesley (**a**) and Olivia's (**b**) graphs depicting accumulated area versus accumulated length

not imagine the nature of change within a given chunk, but she could imagine very tiny chunks and link them together.

In subsequent tasks, the teacher-researcher continued to probe the students' ideas about instantaneous rate. When discussing a growing rectangle the next day, Olivia stated that she believed the instantaneous rate of change of the area at any given amount of length swept would be the same, regardless of how much length had been swept. Wesley agreed, but provided a caveat: "I think so too, but with a triangle, it would be different, because the height is always increasing." This again suggests that he saw the triangle's height at a given length swept as indicative of the rate at that location.

In order to further support the conception that a height can determine the area's rate, the teacher-researcher asked the students to draw a line that was *about* to sweep out an area at a rate of 1.5 cm^2 per cm swept, but had not yet done so. Olivia drew a rectangle with an unspecified length x and a height y of 1.5 cm (Fig. 6a). Olivia still required two extant quantities in order to address the task. She explained, "I did a line with thickness so that we could write down the height and then the length." Wesley drew just a vertical line with a height of 1.5 cm (Fig. 6b). The teacher-researcher asked Wesley, "Does it makes sense to draw just a line as having a rate of change?" Wesley said yes, and then hesitated, amending his answer: "Well, I guess maybe for *instantaneous*, but not for average (rate of change)." He also indicated that his height line could represent a moment in a sweeping journey for any figure: "It could really be any (figure) because maybe as you keep sweeping it out it gets bigger and bigger, or, it just stays the same."

Scaling-continuous covariation and an associated concept of actual infinity could have supported Wesley's identification of a figure's height as a representation of the area's rate of



(a) (b)

Fig. 6 Olivia's (**a**) and Wesley's (**b**) drawings of a line about to sweep out an area at a rate of 1.5 cm^2 per cm change. Wesley saw the height line as a potential rate that could sweep out any amount. His comfort with relying on a *line*, rather than the column that Olivia required, suggests that his ability to imagine smaller and smaller length increments was a completed infinite process that had a resultant state, that of a height line.

5 Discussion

We found evidence that scaling-continuous reasoning afforded productive thinking about constant and instantaneous rates of change. Wesley was able to develop an understanding of a constant rate of change as a rate that represented an equivalence class of ratios. Further, Wesley constructed an understanding of the height of a figure as the area's rate of change. Chunky-continuous covariation can afford an image of re-chunking to very small increment sizes, which can in turn support generalizations about equivalent ratios across different increments. Scaling-continuous covariation, in contrast, enables one to extend such generalizations to any increment size, even the infinitesimal, which can support one's ability to develop and represent rates of change, a foundational idea for algebra and calculus. Thompson and Carlson (2017) noted that "The idea of a function having a nonconstant rate of change is actually constituted by thinking of the function having constant rates of change over small (infinitesimal) intervals of its argument, but different constant rates of change over different infinitesimal intervals of the argument" (p. 452). Olivia's reasoning approached this conception, as she did consider quadratic functions to have constant rates of change over small intervals, but for her, those intervals were not infinitesimal. Scaling-continuous covariation could support this understanding for infinitesimal intervals, as we saw with Wesley's belief that the constantly changing rate of change for triangles and trapezoids could be represented by the figure's height. We also observed in Wesley the less standard belief that a graph of such a function would have an always-changing rate of change even for an infinitesimal interval. As evidenced by Wesley's explanations, his images were a direct outcome of scaling-continuous covariation, in which he could imagine zooming to an infinitesimal scale at the level of actual infinity, with the infinitesimal interval being the final resultant state of an infinite zooming process.

We do not claim that scaling-continuous covariation necessarily preceded and therefore was the only support for Wesley's sensemaking about rates of change. Wesley could have been positioned to develop scaling-continuous covariation because he had already begun to construct equivalence classes of ratios. It could also be the case that Wesley developed a number of these forms of reasoning in tandem, each mutually supporting the other. In addition, we cannot ignore the influence of the task sequence and the teacher-researcher's pedagogical moves in supporting the students' development of chunky-continuous and scaling-continuous covariation. In particular, four salient themes emerged in the role played by the tasks and teacher questioning: (a) a repeated emphasis on continuous motion; (b) directing students' attention to what happens within intervals; (c) encouraging attention to increment size, particularly small and infinitesimal increments, and (d) pushing students to describe and identify rates.

We relied on tasks that used simulations of growing figures that emphasized continuous motion and sweeping actions. The teacher-researcher also emphasized such movement and encouraged the students to first represent rates of change in a non-quantified manner through descriptions and graphs before then transitioning to tasks with measurement. She encouraged the students to imagine change within an increment for a given figure or graph, asking them to describe what occurred within a given chunk. As part of this, the teacher-researcher used tasks that required students to consider rates for different increment sizes, including increments less than 1, and she regularly drew the students' attention to tiny increments and asked them to imagine infinitesimal increments. Finally, both the task sequence and the nature of the teacher-researcher's questioning required students to describe, identify, and ultimately quantify ratios and rates. The students experienced many opportunities to attend to changes in area for corresponding length changes, and the teacher-researcher pushed the students to first describe and then quantify these rates. By emphasizing constant and changing rates and the ways in which they remained invariant across increment sizes, even infinitesimal increments, the nature of instruction in the teaching experiment explicitly supported the development of an image of multiple increment scales, even (for Wesley) infinitely many small increments.

We do not suggest that smooth-continuous covariational reasoning is unimportant for the development of key ideas about function and rate. Indeed, smooth-continuous reasoning is a critical aspect of understanding the mathematics of change, including the ideas of calculus, and we support instructional efforts at all grade levels to develop conceptions of continuous covariation. Instead, we suggest that scaling-continuous covariation offers an additional form of reasoning that may plausibly foster productive understandings to support students' algebraic thinking. Given the potential for this form of reasoning to support key constructs in algebra and calculus, we advocate for additional research to better understand the nature of scaling-continuous variation and covariation and its affordances for productive mathematical thinking.

Funding information The research reported in this paper was supported by the National Science Foundation (grant no. DRL-1419973).

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Journal of Mathematical Behavior 59 (2020) 100795

Contents lists available at ScienceDirect

Journal of Mathematical Behavior

journal homepage: www.elsevier.com/locate/jmathb

A quadratic growth learning trajectory☆

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ARTICLE INFO

Keywords:
Quadratic function
Learning trajectory
Quantitative reasoning
Representational fluency
Instructional supports
DNR

ABSTRACT

This paper introduces a quadratic growth learning trajectory, a series of transitions in students' ways of thinking (WoT) and ways of understanding (WoU) quadratic growth in response to instructional supports emphasizing change in linked quantities. We studied middle grade (ages 12–13) students' conceptions during a small-scale teaching experiment aimed at fostering an understanding of quadratic growth as phenomenon of constantly-changing rate of change. We elaborate the duality, necessity, repeated reasoning framework, and methods of creating learning trajectories. We report five WoT: *Variation*, *Early Coordinated Change*, *Explicitly Quantified Coordinated Change*, *Dependency Relations of Change*, and *Correspondence*. We also articulate instructional supports that engendered transitions across these WoT: teacher moves, norms, and task design features. Our integration of instructional supports and transitions in students' WoT extend current research on quadratic function. A visual metaphor is leveraged to discuss the role of learning trajectories research in unifying research on teaching and learning.

1. Introduction and literature review

1.1. Students' learning of quadratic function

Characterizing and supporting students' meaningful learning of function remains an important and challenging goal of algebra and algebraic thinking across school mathematics (e.g., Ayalon & Wilkie, 2019; Ellis, 2011b; National Governors Association & Council of Chief State School Officers, 2010; Stephens, Ellis, Blanton, & Brizuela, 2017). Non-linear functions in particular can be difficult for students to understand (Ellis & Grinstead, 2008; Lobato, Hohensee, Rhodehamel, & Diamond, 2012; Wilkie, 2019; Zaslavsky, 1997). This study addresses student learning of quadratic functions, an important topic in secondary school, in particular, when students begin to formally develop the algebraic tools to express and represent different functional relationships (Blanton et al., 2018; Stephens, Ellis et al., 2017). There are a host of studies that point to areas of difficulty students experience with understanding quadratic function (e.g., see Wilkie, 2019). In brief, students may struggle to interpret the coefficient of a quadratic and the role of parameters (Dreyfus & Halevi, 1991; Ellis & Grinstead, 2008; Zaslavsky, 1997), have difficulty moving among representations of quadratic functions (Kotsopoulos, 2007; Metcalf, 2007; Moschkovich, Schoenfeld, & Arcavi, 1993), may possess a compartmentalized, procedural view of quadratic functions (Parent, 2015), and often inappropriately generalize from linearity (Ellis & Grinstead, 2008; Schwarz & Hershkowitz, 1999; Zaslavsky, 1997). Despite these well documented difficulties, there are few studies that

☆ Support for this research was provided in part by the U.S. Department of Education IES Research Training Programs in the Education Sciences under grant no. R305B130007, and the National Science Foundation under Award REC 0952415. Any opinions, findings, and conclusions expressed in this material are those of the authors, not the funding agencies.

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https://doi.org/10.1016/j.jmathb.2020.100795
Received 13 October 2019; Received in revised form 3 June 2020; Accepted 8 June 2020

intentionally design instruction to address these challenges to students' learning of quadratic function.

One step toward addressing more robust supports for students' understanding of quadratic functions is to articulate instructional goals and design principles. Ellis' (2011a) study on students' generalization of quadratic growth, for instance, articulated the need to situate students' exploration within visualizable, manipulable contexts in order to identify constantly-changing rates of change of two co-varying quantities. Relatedly, Lobato et al.'s (2012) study elaborated five conceptual learning goals for quadratic function, as well as a set of instructional tasks that could elicit such student understandings. These learning goals included: (a) conceptualize change in dependent quantities, (b) conceive of changes in independent quantities and corresponding sets of changes in dependent quantities, (c) construct a sequence of ratios of change in dependent quantities to change in independent quantities, (d) construct rates of change as a new quantity, and (e) conceive the rate of rate of change of the dependent variable with respect to the independent variable as constant. Both of these studies stressed the importance of supporting students' understanding of quadratic growth within a quantitatively rich situation with a constantly-changing rate of change.

Lobato et al.'S (2012) study provided a viable framework for what students should learn with respect to quadratic function and recommended designing tasks that involve situations with linked quantities such as length-area and time-distance. Related studies have investigated instructional supports for quadratic functions, including principles for task design and the role of multiple representations. For example, in a study of students' intuitions of quadratic growth patterns, Wilkie (2019) explored secondary students' approaches to generalizing quadratic functions from figural growth patterns, and found that "[d]rawing on a visual context such as figural growing patterns might support students in learning conceptually about what actually makes a function quadratic in nature" (p. 16). Wilkie further suggested task design features such as sequencing tasks of the same type, asking students to generalize from visual patterns, and creating quadratic growth patterns by attending to bi-directional links across figures, equations, tables and graphs. In a related study, Selling (2016) argued that communicating with and reasoning about multiple representations supported students' learning of specific aspects of quadratic function such as first and second differences, and explicit and recursive rules. Finally, Rivera and Becker (2016) found that students were able to generalize quadratic growth patterns from a series of figural growth patterns with instructional support. Despite these advances, however, the nature of instructional supports remains underspecified, a gap this study aims to address.

1.2. Research aims

We build on the body of work that advances our understanding of both *what* students are to learn about quadratic functions, and *how* instructional supports may engender that learning. Our aim is to not only expand characterizations of students' learning of quadratic function (*what* students learn), but also to link these characterizations together with mechanisms of learning and instructional supports (*how* students learn). Our design and development inquiries were guided by the following research questions: *(a) How can middle-school students' learning of quadratic growth be characterized and supported? (b) How do goal-directed instructional supports engender that learning?*

To address these questions, we developed a quadratic growth learning trajectory as a representation of transitions in the mathematics of students and the instructional supports that engendered those transitions. We report five Ways of Thinking (WoT) students demonstrated about quadratic growth: *Variation*, *Early Coordinated Change*, *Explicitly Quantified Coordinated Change*, *Dependency Relations of Change*, and *Correspondence*. We also introduce three types of instructional supports that engendered transitions across these ways of thinking: teacher moves, norms, and task design features.

2. Theoretical framework

2.1. Defining a learning trajectory

The construct of a learning trajectory has been discussed in a variety of ways in the literature (Clements & Sarama, 2004; Ellis, Weber, & Lockwood, 2014; Fonger, Stephens et al., 2018; Lobato & Walters, 2017; Simon et al., 2010). Simon's (1995) original construct was for a hypothetical learning trajectory, which consisted of "the learning goal, the learning activities, and the thinking and learning in which students might engage" (p. 133). Some researchers emphasize the evolution of mental concepts as the key aspect of a learning trajectory. For instance, Wilson, Sztajn, and Edgington (2013) defined a learning trajectory as a research-based description of how students' thinking evolves over time from informal to more formal and complex mathematical ideas, Battista (2004) described increased levels of cognitive sophistication through which students progress until they reach formal concepts, and Hackenberg (2013) described a learning trajectory as a model of students' internal conceptions and an account of changes in their schemes and operations. Similarly, Confrey, Maloney, Nguyen, Mojica, and Myers (2009), Panorkou, Maloney, and Confrey (2013) depicted learning trajectories as emphasizing students' refinement of their own conceptual understanding.

Other researchers have emphasized the coordination of learning goals with instructional tasks and activities. Clements and Sarama (2004) described a learning trajectory as consisting of three parts: a mathematical goal, a model of cognition which they called developmental progressions, and instructional tasks providing experiences to support students' movement through the levels. This inclusion of tasks separates their definition from other progressions that document only sequences of students' thinking (e.g., Daro, Mosher, & Corcoran, 2011; National Assessment Governing Board [NAGB], 2008). Sarama (2018) has continued to emphasize the role of curricular tasks, describing a learning trajectory as a "device whose purpose is to support the research-grounded development of a curriculum or other unit of instruction" (p. 72).

Research on learning has largely developed separately from research on teaching (Myers, Sztajn, Wilson, & Edgington, 2015), but

Sarama (2018) cautioned that one cannot discount the role of instruction. Similarly, Confrey et al. (2009) have explicitly acknowledged that conceptual growth, as depicted in a learning trajectory, is influenced by instruction. Simon et al. (2010) have called attention to a paradox in studying learning—that to study learning, instruction must promote the learning one intends to study. Steffe (2004) also directly addressed the role of instruction by acknowledging the importance of accounting for changes in students' concepts and operations "as a result of children's interactive mathematical activity in the situations of learning, and an account of the mathematical interactions that were involved in the changes" (p. 131). The few studies of learning trajectories that directly address instructional actions typically attend to a sequence of learning goals and instructional activities (Fonger, Davis, & Rohwer, 2018, 2018b), instructional practices and activity structures (Fonger, 2018; Stephens, Fonger et al., 2017), the nature of shifts in student conceptions during instructional interventions (Ellis, Ozgur, Kulow, Dogan, & Amidon, 2016), and students' activity and engagement with mathematical tasks (e.g., Simon et al., 2010). Following these approaches, we define a learning trajectory to be an empirically-based model of students' understandings, along with an account of changes in understanding in relation to students' interaction with instructional supports including mathematical tasks, tools and representations, and teacher moves (Ellis et al., 2016).

2.2. DNR-based instruction

We draw on DNR-based instruction (Harel, 2008a, 2008b) in order to inform our instructional design principles for the development, enactment, and analysis of our learning trajectory. DNR stands for the three instructional principles *duality*, *necessity*, and *repeated reasoning*. The duality principle addresses two forms of knowledge, *Ways of Understanding* (WoU) and *Ways of Thinking* (WoT). WoU are students' concepts of specific subject matter, including particular definitions, theorems, proofs, problems, and their solutions (Harel, 2008a). For instance, one WoU about quadratic functions is that quadratic growth is a representation of a relationship between two co-varying quantities in which one quantity varies at a constantly-changing rate of change relative to the other. WoT are broader conceptual tools, such as empirical reasoning, deductive reasoning, heuristics, and beliefs about mathematics (Harel, 2013). For instance, one WoT relevant to students' learning of quadratic functions is a treatment of functional relationships as number patterns, devoid of quantitative referents. Alternatively, one could have a WoT that functions can be investigated as a phenomenon of co-varying quantities. The duality principle states that students develop WoT through the production of WoU, and, conversely, the WoU they produce are afforded and constrained by the WoT they possess (Harel, 2008a). The central objective of the initial content of a learning trajectory, then, must be formulated in terms of both WoU and WoT.

The necessity principle addresses the notion of intellectual need, stating that in order for students to learn the mathematics we intend to teach them, they must experience a need for it (Harel, 2008b). Intellectual need can be engendered through situations that students experience as problematic and that necessitate the creation of new knowledge in order to be resolved. This requires developing problem tasks that students can relate to and become invested in and that necessitate the development of new WoU as a consequence of intellectual engagement with the task, rather than through students' social needs, such as the need to please their teacher, or to achieve a high grade.

Finally, the repeated reasoning principle addresses the importance of ensuring that students internalize, retain, and organize their knowledge (Harel, 2008a). Repeated reasoning is a mechanism for reinforcing desirable WoU and WoT. Repeated reasoning should not, however, be confused with the drill and practice of routine problems. Rather, it is an instructional principle that relies on providing students with sequences of problems that continually call for thinking through puzzling situations and solutions; the problems respond to students' intellectual needs.

2.3. Quantitative reasoning and rates of change

Two common approaches that typically undergird function instruction are the correspondence approach and the coordination or covariation approach. The correspondence approach, which is overwhelmingly prevalent in secondary mathematics curricula and standards in the United States (National Governors Association & Council of Chief State School Officers, 2010; Thompson & Carlson, 2017), emphasizes functional relationships as mappings. A function $y = f(x)$ is defined as a relation between members of two sets, with each value of x mapped to a unique value of y (Smith, 2003). This approach emphasizes the development of closed-form, explicit rules that can be used to analyze and predict function behaviors.

In contrast, Confrey and Smith (1994), Saldanha and Thompson (1998), Smith (2003), Smith and Confrey (1994) and Thompson and Thompson (1992), Thompson (1994), Thompson and Carlson (2017) introduced what they called covariation approaches, although these approaches differ somewhat from one another. Confrey and Smith's (1994) covariation approach emphasizes attention coordinated changes in x- and y-values, in which one can move operationally from y_m to y_{m+1} coordinated with movement from x_m to x_{m+1}. This approach supports engagement with tables and graphs that can be interpreted by students as presenting successive states of variation, supporting the development of an understanding that quantities have sequences of values. We call this way of thinking the *coordination approach*, and note that students can attend to coordinated changes in co-varying quantities even if their images of quantitative situations are static, rather than dynamic.

In a different approach, Saldanha and Thompson (1998) addressed the imagistic foundations that support students' abilities to reason about quantities that vary, either independently or simultaneously. This form of reasoning involves mentally holding a sustained image of two quantities' values simultaneously while attending to how they change in relation to one another (Castillo-Garsow, 2013). From this perspective, covariational thinking entails the coupling of two quantities in order to form a multiplicative object; once this object is formed, a student can attend to either quantity's value with the explicit understanding that at every instance, the other quantity also has a coordinated value. This approach addresses the importance of developing dynamic, rather than

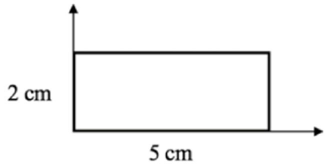


Fig. 1. A diagram of the proportionally growing rectangle context.

static images of quantitative situations, and we call it the covariation approach to distinguish it from the coordination approach described above.

The covariation and coordinated change approaches leverage the notion of a quantity, which Thompson (1994) defined as a person's scheme composed of an object, a quality of the object, an appropriate unit or dimension, and a process for assigning a numerical value to the quality. Thus, a quantity is a conceptual entity, rather than a characteristic that exists in the object itself. Attributes such as length, area, distance, and speed can be conceived as quantities. Quantitative reasoning is the process of reasoning with quantities, their relationships, and associated mathematical operations.

3. Methods

This study is situated in the paradigm of design-based research, in which our goal is to simultaneously engender innovative forms of learning and to study the resulting learning in the context in which it was supported (Cobb & Gravemeijer, 2008; Cobb, diSessa, Lehrer, & Schauble, 2003; Gravemeijer & Cobb, 2006). A design experiment has three phases: *design*, *experiment*, and *analyze*.

3.1. Design

In the design phase we created a hypothetical learning trajectory (Simon, 1995) of goals, potential instructional supports, and hypotheses of students' learning processes informed by the literature on students' learning of quadratic function, DNR-based instruction design principles, and information learned from pre-interviews with each student. Prior research has documented the difficulties in supporting students' meaningful understandings of quadratic function from approaches that emphasize patterning, and representational forms of quadratic functions (see Section 1.1). In contrast, we chose to leverage research and theory on quantitative reasoning and rates of change (see Section 2.3) to ground the design of a hypothetical learning trajectory.

Our goal was to engender students' understanding of quadratic growth as a phenomenon of constantly-changing rate of change among linked quantities (Ellis, 2011a; Saldanha & Thompson, 1998). In order to support this goal, we designed a context with quantities that students could visualize, manipulate, and explore by comparing the lengths, heights, and areas of proportionally-growing rectangles (Ellis, 2011a). These rectangles grew in both length and height but maintained the ratio of length to height (a); thus, the relationship between the height, h, and the area, A, can be expressed as $A = ah^2$ (Fig. 1). Using dynamic geometry software, students could manipulate and measure a rectangle's length, height and area to discern the relationship between these changing quantities.

Within this instructional context, we sought to engender three WoU: (a) the rate of change of a rectangle's area grows at a constantly-changing rate compared to changes in height; (b) given a height, h, the rectangle's area can be determined by $A = ah^2$ where a is the ratio of length to height; and (c) the constantly-changing rate of change of the area, A, is dependent on the change in height (Δh) such that for the ratio of length to height (a), the constantly-changing rate of change is $2a(\Delta h)^2$.[1] We designed a series of tasks that we conjectured would engage students in quantitative reasoning. For example, we asked students to draw several iterations of the proportionally growing rectangle and to create tables relating changes in height and area. Other tasks prompted for generalization of relationships (see Fig. 2).

3.2. Experiment

3.2.1. Setting and participants

The study was situated at a public middle school located in a midsized city in the United States. The participants were 6 eighth-grade students (3 girls and 3 boys) who were enrolled in pre-algebra (3 students), algebra (2 students), and geometry (1 student). The students' teachers identified them as either high, medium, or low based on their assigned mathematics class, as well as on their mathematics grades, attendance, and participation in class. Two students were identified as high, 2 students were identified as medium, and 2 students were identified as low. One student was Indian American, 2 students were Asian American (one of them was an English language learner), and 3 students were Caucasian. Gender-preserving pseudonyms were used for all participants. None of those students had experience with quadratic functions, but all had studied linear functions in their courses.

[1] For an elaboration of the mathematics of this generalized case, see Ellis (2011a).

Make a table for the height, length, and area of the 1x2 rectangle as it grows. Draw each rectangle. How many squares are being added on each time?	Here is a table for the height versus the **AREA** of a rectangle that is growing in proportion. Find the missing quantities. Height — Area 3 — 6.75 4 — 12 5 — 18.75 6 — 27 7 — 36.75 8 — 48 50 — ? h — ?

Fig. 2. Two sample tasks.

Day	Mathematical Topics	Day	Mathematical Topics
1	Measurement and area	9	Justifying the second differences as $2a$
2	Comparing perimeter and area	10	Identifying second differences for tables with different Δh values
3	Identifying first and second differences in tables	11	Connecting equations, tables, and graphs
4	Connecting first and second differences to area	12	Graphing parabolas
5	Identifying height : length ratios and creating $y = ax^2$ equations	13	Graphing first and second differences
6	Creating generalizations about second differences	14	Creating $y = ax^2 + c$ equations and graphs
7	Justifying generalizations about second differences	15	Summarizing generalizations
8	Creating $y = ax^2$ equations from tables and identifying the second differences as $2a$		

Fig. 3. Overview of the TE sessions (reproduced from Ellis, 2011a, p. 313).

3.2.2. Teaching experiment

We conducted a 15-session videoed teaching experiment [TE] (Cobb & Steffe, 1983), with each session lasting 1 h. The TE setting allowed for the creation and testing of hypotheses in real time while engaging in teaching actions. This meant that the mathematical topics for the entire set of sessions were not predetermined but instead were created and revised on a daily basis in response to hypothesized models about the students' mathematics. Fig. 3 provides a brief overview of the topics addressed in the TE. Task design was also an ongoing part of the experiment. For example, as we learned new information about student learning processes, we designed new tasks to support the students' intellectual need to elaborate and justify relationships that held for *all* proportionally growing rectangles.

The second author was the teacher-researcher (TR), and two project members observed each teaching session. The TR fostered a learning environment that encouraged students to make and test conjectures, to make predictions and generalizations, and to explicitly attend to quantities and their relationships. During TE activities, the students worked individually, in pairs, and in small groups, before then discussing their ideas with the entire group. The project members operated two video cameras during the teaching sessions to capture both the whole-group discussions and the small group interactions. The project team met after each session to debrief, discuss what had occurred during the session, and design new tasks or revisions to planned tasks.

3.3. Analyze

Our goal in creating a learning trajectory was to establish rich descriptions of the nature of learning in the context of instructional supports (cf. Cobb & Gravemeijer, 2008). To attain this goal, we conducted several rounds of data analysis with differing foci. Our primary focus was to articulate change in the mathematics of students[2] ; our secondary focus was to contextualize and explain these changes with respect to instructional supports. All sessions of the TE were transcribed; then, all transcripts were enhanced to include verbal utterances, images of the drawings on the board, student's written predictions prior to discussion, and descriptions of the students' gestures. To analyze the data, we applied the constant comparative method (Strauss & Corbin, 1990) and axial coding (Strauss, 1987) to develop and discern relationships among codes.

In the first round of coding, all three authors independently coded the first eight sessions and then met to discuss and reconcile

[2] We distinguish between first-order knowledge (one's own knowledge) and second-order knowledge (a model of another's knowledge) (cf. Steffe, von Glasersfeld, Richards, & Cobb, 1983). In this paper we aim to build second-order models of students' first order knowledge. Said otherwise, *students' mathematics* entails "a students' first-order mathematical knowledge" while the *mathematics of students* entails "explanatory models of students' mathematics" (Steffe, n.d., p. 7). Thus, in our focus on building models of students' mathematics, we are characterizing the mathematics of students.

their coding decisions. During this phase, we established an initial framework of emergent categories and subcategories of relevant student concepts. In the second phase, we applied our coding framework to sessions 9–15, again meeting to reconcile our coding after each session. These reconciliations resulted in modifying code descriptions, adding new codes, and collapsing existing codes. After these two rounds, we reached a stable coding scheme and then re-coded all sessions with the final scheme, which included specific WoU and broader WoT. As a note, Harel used the terms WoU and WoT to distinguish specific mathematical ideas from broader heuristics and beliefs. In contrast, we have adapted these terms to distinguish clusters of concepts as WoT, and then the specific concepts within each cluster as WoU. In order to distinguish how we have adapted Harel's term "WoT" to identify these clusters, we will hereafter refer to them by number: *WoT1*, *WoT2*, etc. These numbered WoT are distinguished from the more general WoT that are not part of the learning trajectory, but are instead heuristics and beliefs, such as attending to patterns.

In the final phase of coding, we identified transitions in students' WoT and how those shifts occurred in response to instructional supports. We identified an *instructional support* by analyzing the data corpus for evidence of task features, teacher moves, and other features that appeared to engender students' WoT and WoU. While the instructional supports were emergent, they were also informed by our knowledge of the existing literature on teacher moves (e.g., Bishop, Hardison, & Przybyla-Kuchek, 2016; Franke et al., 2009; Peterson et al., 2017), *norms* (Cobb & Yackel, 1996; Stephan, 2014), and *tasks* (Bieda & Nathan, 2009; McCallum, 2019; Stein & Smith, 1998).

4. Results: the quadratic growth learning trajectory

In this section we introduce models of students' WoT and WoU about quadratic growth, the transitions that occurred from one WoT to the next, and the instructional supports that engendered those transitions. We identified five WoT: *WoT1 Variation*, *WoT2 Early Coordinated Change*, *WoT3 Explicitly Quantified Coordinated Change*, *WoT4 Dependency Relations of Change*, and *WoT5 Correspondence*. Each of these five WoT includes a number of specific WoU. We also found three types of *instructional supports*, which we categorize into (a) teacher moves, (b) norms, and (c) task design features.

A *teacher move* is what the TR does in response to students' mathematical thinking, or to elicit new instances of mathematical thinking when engaging an individual student, a small group, or the whole class (cf. Peterson et al., 2017). We found these teacher moves to fall under three broad design heuristics: quantitative reasoning, representational fluency, and generalization. *Norms* are the expectations that the teacher and students have for each other that are present during mathematical discussion or student engagement with mathematical tasks (cf. Cobb & Yackel, 1996). We found the norms to cluster around two design heuristics—quantitative reasoning and representational fluency—as well as social norms for mathematical discourse in classroom interaction. A *task* is statement of a mathematical problem or set of problems that focuses students' attention on a particular mathematical idea or provides an opportunity for students to engage in a particular way of thinking (cf. Stein & Smith, 1998). We found several *task design features* and characterized these as falling into three types: doing mathematics, far prediction and generalization, and repeated reasoning. Sample tasks are given in the methods section and the forthcoming results subsections.

The learning trajectory for quadratic growth is introduced in Table 1. The learning trajectory includes the mathematics learning goal, the mathematics of students—both *WoT1* through *WoT5* and their subsequent *WoU*—and the instructional supports that engendered transitions in the mathematics of students. We caution the reader *not* to interpret the learning trajectory as a set of understandings that were predetermined, nor as a list of understandings and instructional supports that occurred in a linear order. Instead, this learning trajectory constitutes a model characterizing how the students learned to organize their WoT and related WoU in an intentionally designed, supportive instructional context guided by a conceptual learning goal. Fig. 4 introduces a visualization of the set of instructional supports unified with the WoT and WoU. This depicts the quadratic growth learning trajectory as a dynamic model of transitions.

In the following sections we elaborate the four major transitions the students experienced from one WoT to the next. Each section addresses the relevant WoT with select examples of WoU and the instructional supports that engendered the students' transitions.

4.1. Transition I: from variation to early coordinated change

The first transition in the learning trajectory is from *WoT1 Variation,* to *WoT2 Early Coordinated Change*. The students' initial attention to variation was such that they did not coordinate change across quantities. Instead, they only attended to variation in one quantity at a time, or, they attended to two types of variation but only as a sequence of disconnected changes.

4.1.1. WoT1 variation

Initial tasks for the quadratic growth situation incremented a proportionally growing rectangle by 1 cm in height, and we then began to introduce tasks in which the rectangles' height or length values grew by greater than 1 cm. The students created their own data tables to keep track of the growth in height, length, and area. For example, given a growing square, Jim created a table relating the square's length and width to its area (Fig. 5). When the TR asked Jim to discuss his table, Jim focused solely on describing the area's second differences of 18 cm^2 as "going up by 18 s" without attending to how the square's height (or length) grew. We call this *WoU1.1 Single-Quantity Variation* because the students noticed and described changes in one quantity's magnitude without attending to the other quantity or changes in its magnitude. We note that it is likely that the students did not even implicitly attend to changes in the other quantity's magnitude because they did not experience an intellectual need to do so, given that their tables were well-ordered.

Another observed WoU within *WoT1* is *WoU1.2 Uncoordinated Variation*, in which students began attending to variation across

Table 1
A Quadratic Growth Learning Trajectory.

Conceptual Learning Goal			
Support students' understanding of quadratic growth as a relationship between quantities such that the dependent quantity y has a constantly-changing rate of change with respect to the independent quantity x. In symbols, for a quadratic function $f : x \rightarrow y$, if $\Delta\Delta y = d$, then $y = d/(2(\Delta x)^2) \bullet x^2$			
Students' Ways of Thinking (WoT), Ways of Understanding (WoU)	Instructional Supports		
Initial *WoT1 Variation* • WoU1.1 Single-Quantity Variation • WoU1.2 Uncoordinated Variation	***Teacher Moves***	***Norms***	***Task Design Features***
Transition I *WoT2 Early Coordinated Change* • WoU2.1a,b Implicit Coordinated Change • WoU2.2 Qualitative Coordinated Change	***Quantitative Reasoning:*** • Ask students to explicitly attend to how quantities change together as expressed in diagrams, words, and tables; • Explicitly draw attention to and identify quantities; and • Press for quantitatively-based justifications.	***Quantitative Reasoning:*** • Be explicit about the amount of change for each of the relevant quantities; • Attend to linked quantities as relationships of explicit coordinated change; and • Make and test predictions about generalized relationships among linked quantities.	***Repeated Reasoning:*** • Investigate a quadratic growth situation where change in independent quantity (Δx) is greater than 1; • Investigate a quadratic growth situation where change in independent quantity (Δx) is less than 1; • Investigate a collection of tables for quadratic growth where the change in the independent quantity (Δx) is greater than 1 and varied across the set; and • Sequence tasks of the same type to allow students to test conjectures about generalized relationships.
Transition II *WoT3 Explicitly Quantified Coordinated Change* • WoU3.1a,b Single-Unit Explicit Coordination • WoU3.2a,b Multiple-Unit Explicit Coordination • WoU3.3a,b Partial-Unit Explicit Coordination	***Representational Fluency:*** • Ask students to create their own tables of two or more quantities changing together; • Prompt students to connect different representations; • Encourage students to create drawings or diagrams and to name quantities; and • Encourage students to create graphs.	***Representational Fluency:*** • Create tables and drawings to express relationships in quadratic growth contexts.	***Far Prediction and Generalization:*** • Predict the changing rate of change in a quadratic growth situation from a table of quantities; • Elicit conjectures about generalized relationships among linked quantities for a quadratic growth context; • Make far predictions of independent and dependent quantities for a given quadratic growth pattern; and • Elicit generalizations of relationships between independent and dependent quantities using variables.
Transition III *WoT4 Dependency Relations of Change* • WoU4.1 Recognition that Change in One Quantity Determines Change in the Other • WoU4.2 Identification of How Change in One Quantity Determines Change in the Other • WoU4.3 Translations of Dependency Relations of Change to Correspondence Rules	***Generalization:*** • Prompt students to make conjectures and test their conjectures in a new task context; and • Ask students to generalize a mathematical relationship they identified.	***Mathematical Discourse:*** • Multiple student responses are shared for a single question or task; and • Students respond to and build on each other's ideas.	***Doing Mathematics:*** • Encourage thinking through puzzling situations and solutions (e.g., given an un-specified quadratic growth pattern); and • Quadratic growth task allows for multiple different ways of interpreting and explaining one's reasoning.
Transition IV *WoT5 Correspondence* • WoU5.1a,b Correspondence Relations Between Independent and Dependent Quantities			

multiple quantities as isolated patterns. This was particularly prominent in far prediction tasks for tables in which the height was incremented by more than 1 cm. For example, given the table in Fig. 6, Ally conceived of the difference in length as "going up by eight" and also noted that "It's going up by 2 s (in height)," yet did not attend to how the two quantities changed together.

In Table 2 we summarize the two component WoU for *WoT1 Variation*. Notice in this table that each instructional support is identified for each WoU, one by one. All of the students entered the TE attending to changes in the values of one or more quantities; we saw this as evidence of a general WoT that valued pattern-seeking. The students were adept at finding and identifying many different patterns, particularly patterns within one quantity. This WoT supported the development of the students' initial *WoT1* of attending to single-quantity or disconnected variation. Thus, in task situations such as that given in Fig. 6, coordinated variation did

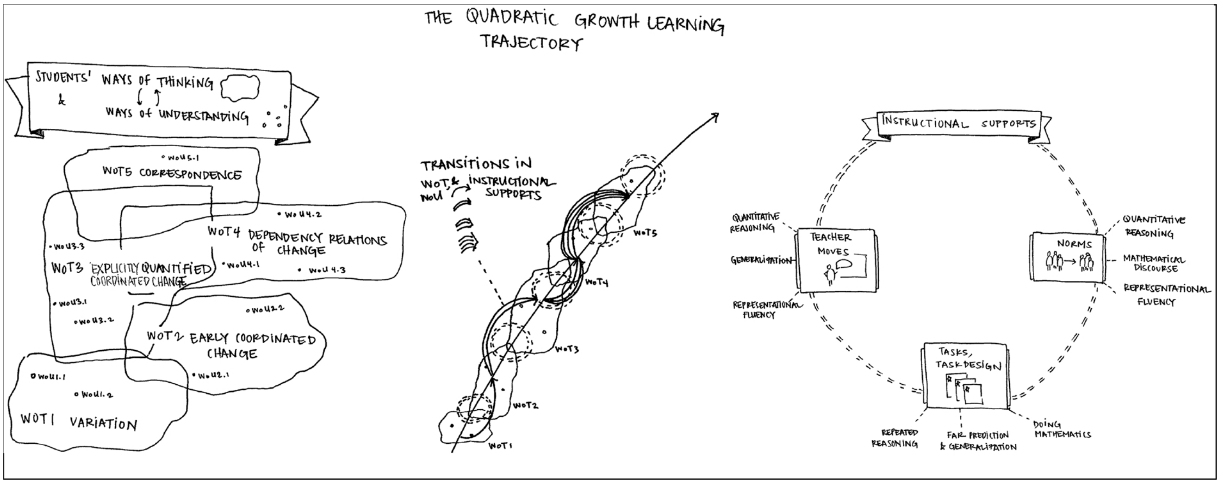


Fig. 4. A visualization of the quadratic growth learning trajectory as a dynamic model of transitions in students' WoT and WoU together with an integrated set of instructional supports.

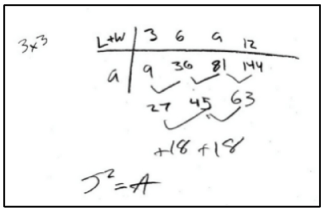


Fig. 5. Jim attended to single-quantity variation.

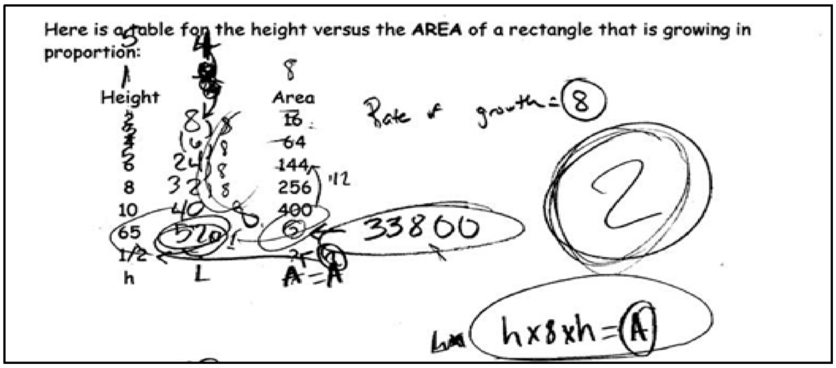


Fig. 6. Ally's uncoordinated variation of quantities.

Table 2
WoT1 Variation, Related WoU, and Instructional Supports.

WoT1 Variation (Conceiving of single or multi-quantity variation without coordination)			
WoU	Definition	Data Example	Instructional Support
WoU1.1 Single-Quantity Variation	Student attends to a change in one quantity without coordinating this change with any other quantity change.	Jim: "This one is going up by 18 s."	*Task Design Feature*: Ask students to investigate a quadratic growth situation given an initial independent quantity (x) greater than 1.
WoU1.2 Uncoordinated Variation	Student attends to a change in more than one quantity without coordinating simultaneous change; variation is treated as isolated patterns or sequences.	Ally: "I figured out it was going up by eight (in length)" and "It's going up by 2 s (in height)."	*Teacher Move:* Ask students to explicitly attend to how quantities change together as expressed in diagrams, words, and tables.

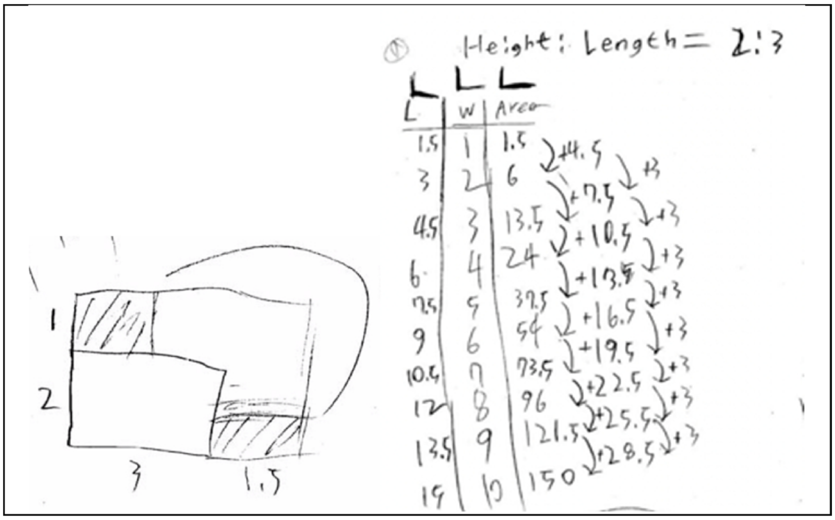


Fig. 7. Daeshim's drawing and table of the proportionally growing 2 cm by 3 cm rectangle.

not occur spontaneously for students. We found that several instructional supports were necessary for students to develop attention to coordinating changes in the values of two or more quantities, as we describe in the next section.

4.1.2. WoT2 early coordinated change

When students began to attend to coordinated change in the values of two or more co-varying quantities, we found that their initial conceptions of these changes were either qualitative (unquantified) or implicit for at least one of the quantities. We call this *WoT2 Early Coordinated Change*. We identified two main WoU for *WoT2*: *Implicit Coordinated Change (WoU2.1a,* and *WoU2.1b)* and *Qualitative Coordinated Change* (*WoU2.2*). Additionally, we found five instructional supports for developing *WoT2*: *teacher moves* of (1) asking students to explicitly attend to how quantities change together as expressed in diagrams, words, and tables, (2) press for quantitatively-based justifications, (3) prompts to connect different representations; a series of *tasks* with task design feature that afforded (4) investigation of a quadratic growth situation where change in independent quantity (Δx) is greater than 1, and (5) *tasks* that encouraged thinking through puzzling situations and solutions (e.g., with an un-specified growth pattern). As we exemplify below, we found certain instructional supports to engender particular WoU. Moreover, some instructional supports engendered more than one WoU within *WoT2*.

4.1.2.1. WoU2.1 implicit coordinated change. During the TE, the TR encouraged the students to describe how the values of the growing rectangle's quantities height, length, and area changed, as well as to explicitly connect how these quantities were interpreted in tables, figures, and words. For example, students investigated a proportionally growing rectangle that maintained a 2:3 ratio of height to length on a Geometer's Sketchpad © file that could be dragged to show dynamic growth. Daeshim's drawing and table are given in Fig. 7. Prompted to explain the table, Jim described "then [the] branch off of those is that they all go up by 3" (see the +3 on the right side of the table in Fig. 7). The TR pressed students to explain this finding:

TR: Can you anticipate what I'm going to ask you now?

Jim: Where does the 3 come from?

TR: Where does the 3 come from? What does the 3 have to do with the picture?

Anna: Uh, this is just a guess, but uh, when it goes up by 3 s, like it did with the 2 s, like every time it grows it adds 3.

In this excerpt, the TR pressed the students to give a quantitatively based explanation for their finding, and Anna coordinated the constantly-changing difference in change in area (3 more squares) implicitly with a change in height, stating, "*every time* it grows, it adds 3." We take this as evidence of *WoU2.1b Implicit Coordination of Second Differences with Change in Another Quantity*. Note that *WoU2.1 Implicit Coordinated Change* includes two sub-levels; *WoU2.1a* focuses on implicit coordinated change in two quantities, while *WoU2.1b* focuses on coordinated change of second differences with change in another quantity (see Table 3).

4.1.2.2. WoU2.2 qualitative coordinated change. The second WoU for *WoT2* is *WoU2.2 Qualitative Coordinated Change*. An example of this WoU occurred when the students examined a "Mystery Table" (see Fig. 8), in which the area function was unknown. After the students worked on the task, the TR invited students to imagine how the height and length of the rectangle was growing based on their reading of the table.

TR: Is the rectangle growing in one direction only or do you think it's growing in both directions?

Table 3
WoT2 Early Coordinated Change, Related WoU, and Instructional Supports.

WoT2 Early Coordinated Change (Conceiving of multi-quantity variation without explicit quantification of both quantities)			
WoU	Definition	Data Example	Instructional Support
WoU2.1a Implicit Coordinated Change in Two or More Quantities. *WoU2.1b Implicit Coordination of Second Differences with Change in Another Quantity.*	Students attend to growth in both quantities together, but the magnitude of change remains implicit for one or both quantities. Students coordinate the difference in the rate of growth of the area, but the magnitude of the change remains implicit for one or both quantities.	Jim: "How many new squares it's gaining every time it grows." Anna: "like every time it grows it adds 3 [more squares]"	*Teacher Moves:* ● Ask students to explicitly attend to how quantities change together as expressed in diagrams, words, and tables; and ● Press for quantitatively-based justifications. *Task Design Feature:* ● Investigate a quadratic growth situation where change in independent quantity (Δx) is greater than 1.
WoU2.2 Qualitative Coordinated Change	Student links the change in two or more quantities, understanding that they change together, without quantifying the change.	Daeshim: "Well, it's the length times height. If length were growing, area will be bigger".	*Teacher Moves:* ● Ask students to explicitly attend to how quantities change together as expressed in diagrams, words, and tables; and ● Prompt students to connect different representations. *Task Design Feature:* ● Encourage thinking through puzzling situations and solutions (e.g., given an un-specified growth pattern).

Here is a table for a rectangle that's growing in a way I'm keeping secret

Height	Area
2	5
5	12.5
7	17.5
8	20
10	25
1	______
$\frac{1}{2}$	______
h	______

1) Is the rectangle growing in one direction only, or in both directions? How do you know?

2) What type of graph do you think this will be? Make a prediction.

Fig. 8. A Mystery Table Task.

Bianca: Both.

TR: Both? How come?

Jim: Because the height and length are changing in numbers.

Jim said that both the height and length were changing, but he did not quantify the change. This example illustrates that as the students began to attend to changes in two or more quantities simultaneously, they would often describe these changes without quantifying the nature of the change. In another example, Daeshim said, "Well, it's the length times height. If length were growing, area will be bigger." The instructional supports that encouraged attention to simultaneous (albeit qualitative) change included a teacher move of eliciting students' descriptions for how the rectangle grew, as well as the use of tasks that relied on "Mystery Tables". These tables provided students with height and area values and challenged them to determine the nature of the rectangle's growth, necessitating attention to how both quantities grew together. Another teacher move that encouraged the development of *WoT2* was a prompt to connect different representations. For instance, the TR encouraged the students to use the data provided in tables to draw rectangles with labelled quantities of height and length.

As summarized in Table 3, students' transition to *WoT2 Early Coordinated Change* was characterized by either *Implicit Coordinated Change* (*WoU2.1a* or *WoU2.1b*) or a *Qualitative Coordinated Change* (*WoU2.2*). We found the same collection of instructional supports to engender both *WoU2.1a* and *WoU2.1b*, hence the grouping across these rows in Table 3. Instructional supports included teacher moves such as prompting students to identify and articulate changes in quantities across figures, tables, graphs, as well as tasks that: (a) varied the initial value of the height, and (b) encouraged attention to two quantities changing together (e.g., the Mystery Table).

4.2. Transition II: establishing explicitly quantified coordinated change

The second major transition in the learning trajectory occurred when the students developed the *WoT3 Explicitly Quantified Coordinated Change*. By explicit quantified coordination, we mean that the students identified the amounts of change in each of the quantities as they coordinated changes in both quantities together. This WoT includes three main WoU: (a) *Single-Unit Explicit Coordination* (*WoU3.1a* and *WoU3.1b*), (b) *Multiple-Unit Explicit Coordination* (*WoU3.2a* and *WoU3.2b*), and (c) *Partial-Unit Explicit Coordination* (*WoU3.3a* and *WoU3.3b*). We found seven instructional supports that fostered these WoU. They included the *teacher moves* of: (1) explicitly drawing attention to and identifying quantities, (2) prompting students to create their own tables of two or more quantities that changed together, (3) pressing for quantitatively-based justifications, and (4) encouraging students to create drawings, diagrams, or graphs and name quantities. The instructional supports also included *tasks* requiring students to: (5) predict the constantly-changing rate of change, and (6) investigate tables of linked quantities with increments less than 1 for the independent variable. A *norm* was also established: (7) be explicit about the magnitude of change for each of the relevant quantities. We elaborate the specific links between this set of instructional supports and the particular WoU these instructional supports engendered in the following subsections.

4.2.1. WoU3.1 single-unit explicit coordination

As discussed above, when the students initially began to coordinate changes in co-varying quantities, they did so without explicitly attending to the amount of change. In an attempt to encourage the students to explicitly attend to both the relevant quantities and the nature of their coordinated changes, the TR began to model this type of attention focusing through the teacher moves of drawing figures and naming quantities. She also asked the students to create their own tables to compare the rectangle's length to its area. The manner in which the table was organized (what values to begin with and what the amount of increase should be from one entry to the next) was left open to the students. In creating their own tables and making decisions about both initial values and amounts of increase, the students gained facility with explicitly attending to coordinated changes.

Recall Daeshim's table and figure for a proportionally growing 2 cm by 3 cm rectangle (Fig. 7). As the students created drawings and tables, Bianca said, "I found that the length is 1.5 times the height." The TR asked her to explain on the board with a picture. Bianca drew a growing rectangle similar to that shown in Daeshim's drawing on the left hand side of Fig. 7. She then explained, "If you increase it by 1 (adds on to her rectangle), then you've got, now it's 3 by one point, er, 4.5. So 3 times 1.5 equals 4.5. It just keeps going like that." As Bianca grew the rectangle in her drawing, she was forced to coordinate growth in height with growth in length, realizing that for every 1-cm growth in height, there was a corresponding 1.5-cm growth in length: "For every 1 you add it up, then you add 1.5 across." This is evidence of *WoU3.1a Single-Unit Explicit Coordinated Change in Two Quantities*.

Once the students noticed this pattern, the TR asked them to explain why the area grew by a constantly-changing rate of change that increased by 3 cm^2 per each 1-cm increase in height: "Where does the 3 come from? What does the 3 have to do with the picture?" This press for a quantitatively-based justification served as a support to emphasize attention on how the quantities changed together. Daeshim suggested, "1.5 plus 1.5 equals 3":

TR: And where are you getting the 1.5 and the 1.5?

Daeshim: It's, if width increases 1, then length increases 1.5.

Daeshim suspected that the constantly-changing rate of the change of 3 cm^2 in area per 1 cm in height was related to the ratio of length to height, 1.5:1, but was not yet sure why.

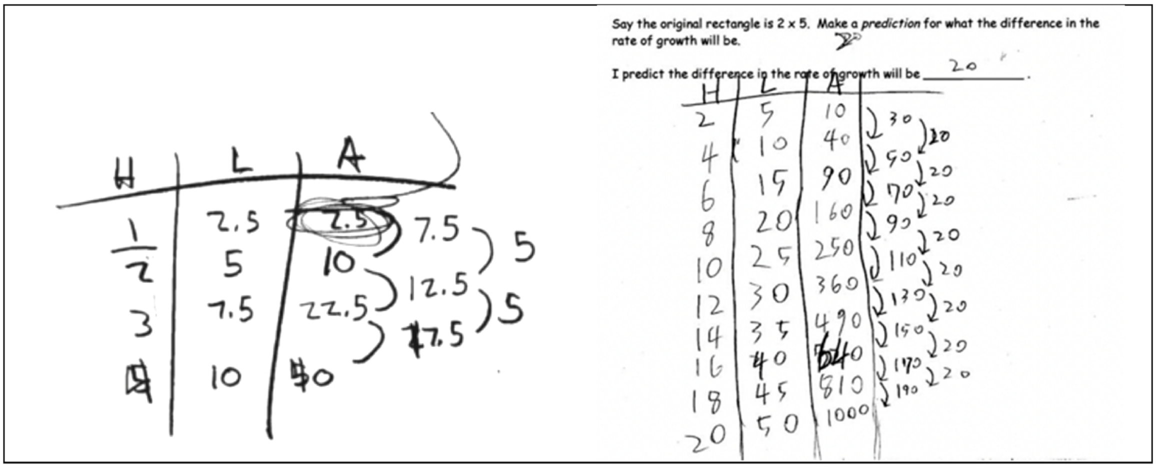
4.2.2. WoU3.2 multiple-unit explicit coordination

Over time it became clear that some students preferred to always increase height values by 1 cm, regardless of the original dimensions of the rectangle. The TR therefore devised a task in which the students had to first predict the changing rate of change for the growth in area of a 2-cm by 5-cm rectangle, and then make their own tables to test their predictions. She suspected that some students might increment the tables by a unit of 2 cm for the height, while others would increment by 1 cm. In addition, by not specifying the amount of increase for the height for which the students' prediction should occur, the TR anticipated that the students would identify different rates of change for the area.

Some students predicted that the changing rate of change for the area would be 5 cm^2, others predicted 10 cm^2, and others predicted 20 cm^2. As anticipated, the students did not initially specify for what change in height their prediction referred to. Additionally, when testing their predictions, some students created tables with a height increase of 1 cm, while others created tables with a height increase of 2 cm (Fig. 9). This difference in prediction led to a disagreement among the students about whether the constantly-changing difference in the change in area should be 5 cm^2 or 20 cm^2. Jim claimed "It's 5." Bianca exclaimed, "You guys! It's neither! It's, it's, it's 20!" Daeshim also found the constantly-changing difference of change of area to be 20 cm^2, which bolstered Bianca's confidence that she was correct and Jim was not.

Bianca: Daeshim is right. And mine is right.

Jim: No, but, I'm doing 1. I'm going by 1.



(a) (b)

Fig. 9. (a) Jim's and (b) Daeshim's tables for a 2cm × 5cm growing rectangle prediction task.

Tai: No, but his is going by 1 too.

Jim: *He's* going by 2's. But *I'm* going by 1's.

Jim introduced the need to attend to the change in height values in the students' respective tables. The students then realized that the organization of their tables, particularly the magnitude of height increases, was relevant for determining the constantly-changing rate of change in the area. They concluded that both answers were correct, depending on the choice of height increase. Tai explained, "Yeah, 20 and 5 would both work because we're going up by 2's, and they're going up by 1's." Tai's articulation of a constantly-changing rate of change in area of 20 cm^2 per change in height by 2 cm is evidence of *WoU3.2b Explicit Coordination of Second Differences with a Multiple-Unit Change in Another Quantity.*[3] The teacher move of asking the students to create their own tables, and task design feature to predict the constantly-changing rate of change, both supported the students in attending to coordinating changing quantities for magnitude increases other than 1. In addition to creating tables, the TR's continued prompts to create drawings of the quantities in the growing rectangle situation also encouraged an explicit focus on coordinating changes in quantities for multiple-unit iterations.

The students became accustomed to identifying different changes in height as coordinated with changes in length and constantly-changing rates of change in area. The students typically did this as a response to the regular occurrence of needing to reconcile different area rates that emerged as a consequence of different table organizations. As a way to navigate these differences across the students' work, a norm emerged that it was critical to become explicit about the amount of change for each of the relevant quantities.

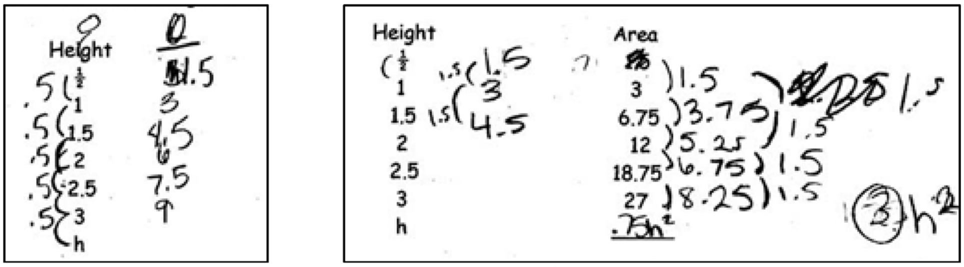
4.2.3. WoU3.3 partial-unit explicit coordination

A final WoU in *WoT3* entails coordination of partial units (less than 1). In the following example, the TR presented a task comparing height and area in a table, with the magnitude of change for height being 0.5 cm. In attempting to determine the changes in area, the students found the corresponding length values for each height value (Fig. 10a), determining that the length increased by 1.5 cm for every ½-cm increase in height. They also found that the constantly-changing rate of change for the area was 1.5 cm^2 for every ½-cm increase in height (Fig. 10b). In Jim's words, "So, like these *two*, it's 1.5 [indicates an increase across two rows, rather than one row]. These two it's 1.5. It's going up 3."

By this point in the TE, the students had developed a norm of stating the constantly-changing rate of change in area for a 1-unit increase in height. Jim realized that claiming 1.5 cm^2 was misleading, because that was not for a 1-cm increase, but rather for a ½-cm increase. Thus, he wanted to express the constantly-changing second differences in area as 3 cm^2, because he recognized that the increase of 1.5 cm^2 was coordinated with ½-cm rather than 1-cm increments in height. We call this *WoU3.3b Explicit Coordination of Second Differences with a Partial-Unit Change in Another Quantity*.

Introducing a table with an increment less than 1 for the height did support the students' partial-unit coordination of height with length and height with changes in area. However, we posit that this instructional support would not have been effective had it not occurred after the establishment of a norm that it was necessary to explicitly attend to the magnitude of height increases. That norm began when the students encountered conflicting results from creating their own tables with different magnitudes of change, and it

[3] We would like to draw attention to a subtle nuance in the language we chose to describe the students' WoU here. First, we describe Daeshim's attention to a "constantly-changing difference of change in area." We then shifted our language to describe Tai's "constantly-changing rate of change in area per height." Instructionally, we aimed to support students' attention to construct constantly-changing rates of change of area with respect to height (and with respect to the ratio of height to length), as described in Section 3.1. However, in analyzing the students' WoT and WoU, we take care to characterize a model of the students' mathematics in our descriptions.



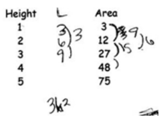
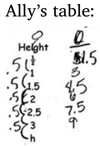
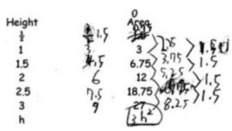
(a) (b)

Fig. 10. (a) Ally's and (b) Bianca's tables for a growing rectangle growing by ½-cm in height.

continued as the TR deliberately introduced rectangles with configurations that would encourage students to increment height values by amounts other than 1.

Table 4 summarizes the WoU under *WoT3 Explicitly Quantified Coordinated Change*. We found these WoU emerged in response to several instructional supports including: *teacher moves* such as explicitly drawing attention to and identifying quantities in tables,

Table 4
WoT3 Explicitly Quantified Coordinated Change, Related WoU, and Instructional Supports.

WoT3 Explicitly Quantified Coordinated Change (Conceiving of multi-quantity variation with explicit quantification of both quantities)			
WoU	Definition	Data Example	Instructional Support
WoU3.1a Single-Unit Explicit Coordination of Change in Two Quantities	Student coordinates the change in two or more quantities together, and also quantifies the amount of both changes. In this case, change in one quantity is a unit change.	Jim: "The way it kinda looks. Like, it's going up 1 over 2 every time. So it looks like it."	*Teacher Moves:* • Explicitly draw attention to and identify quantities; • Ask students to create their own tables of two or more quantities changing together; • Press for quantitatively-based justifications; and • encourage students to create drawings or diagrams and name quantities.
WoU3.1b Explicit Coordination of Second Differences with a Single-Unit Change in Another Quantity	A single-unit change in one quantity is coordinated with the difference in the rate of growth of the area.	Tai: "Yeah, 20 and 5 would both work (for second difference in area) because we're going up by 2 s (in height) and they're going up by 1 s (in height)." *Note: this is also an example of WoU3.2b.*	
WoU3.2a Multiple-Unit Explicit Coordination of Two Quantities	Student coordinates the amount of change for two quantities for multiple-unit changes in one of the quantities.	Bianca: "Each time, it's going up by eight [in length], and each of these [height] is going up by two. The ratio is two to eight, or one to four."	*Teacher Moves:* • Prompt students to create their own tables of two or more quantities changing together; and • Encourage students to create drawings or diagrams and to name quantities. *Task Design Feature:* • Predict the changing rate of change from a table of quantities representing quadratic growth. *Norm:* • Be explicit about the amount of change for each of the relevant quantities.
WoU3.2b Explicit Coordination of Second Differences with a Multiple-Unit Change in Another Quantity	A multiple-unit change in one quantity is coordinated with the difference in the rate of growth of the area.	Bianca: You add 3 to the length and 6 to the area each time. 	
WoU3.3a. Partial-Unit Explicit Coordination of Two Quantities	Student coordinates the amount of change for two quantities for partial-unit change in one of the quantities.	Ally's table: 	*Teacher Moves:* • Ask students to create their own tables of two or more quantities changing together; and • Encourage students to create drawings, diagrams, or graphs. *Norm:* • Be explicit about the amount of change for each of the relevant quantities. *Task Design Feature*: • Investigate a quadratic growth situation where change in independent quantity (Δx) is less than 1.
WoU3.3b. Explicit Coordination of Second Differences with a Partial-Unit Change in Another Quantity	A partial-unit change in one quantity is coordinated with the difference in the rate of growth of the area.	Daeshim's table: 	

drawings, or diagrams students created; *norms* such as being explicit about the amount of change for each of the relevant quantities; and *task design features*. Across these ways of understanding, students quantified the change in two or more linked quantities (*WoU3.1a*, *WoU3.2a*, *WoU3.3a*), and explicitly quantified coordination of second differences in the dependent variable with another quantity (*WoU3.1b*, *WoU3.2b*, *WoU3.3b*). We found the instructional supports for *WoU3.1a* and *WoU3.1b* were the same (likewise for *WoU3.2a* and *WoU3.2b*; and *WoU3.3a* and *WoU3.3b*).

4.3. Transition III: establishing dependency relations of change

The third transition occurred when the students began to articulate *WoT4 Dependency Relations of Change* relating height, length, and area. Dependency relations of change means that the students recognized and identified how changes in linked quantities were dependent on other quantities or changes in those quantities. *WoT4* includes three WoU: (a) *Recognition that Change in One Quantity Determines Change in the Other* (*WoU4.1*), (b) *Identification* of *How Change in One Quantity Determines Change in the Other* (*WoU4.2*), *and* (c) *Translation of Dependency Relations of Change to Correspondence Rules* (*WoU4.3*). We found nine instructional supports that fostered these WoU. The *teacher moves* included: (1) pressing for quantitatively-based justifications. The *norms* included: (2) being explicit about the amount of change in relevant quantities, (3) students responding to and building on one another's ideas, and (4) having multiple student responses shared for the same task. *Tasks* were designed to include: (5) quadratic growth situations that allow for multiple different ways of interpreting and explaining one's reasoning, (6) prediction of the constantly-changing rate of change in the dependent quantity from a table of quantities, (7) investigation of a quadratic growth situation given an initial independent quantity (x) greater than 1, (8) investigation of a collection of tables for quadratic growth where the change in the independent quantity is greater than 1 ($\Delta x > 1$), and varied across the set, and (9) making conjectures about generalized relationships among linked quantities for a quadratic growth context. In the following sections we provide excerpts to illustrate how these instructional supports encouraged the development of the three WoU.

4.3.1. WoU4.1 recognition that change in one quantity determines change in the other

The TR had been engendering norms in the classroom in which multiple student responses were elicited for a single question or task, and students were encouraged to respond and build on each other's ideas. Moreover, tasks allowed multiple different ways of interpreting and explaining one's reasoning. In this context, the students started to recognize and identify how changes in one quantity determined changes in the other. In one example, the students were pressed to give a quantitatively-based argument for why the constantly-changing rate of change of the area for a 1 cm by 2 cm rectangle was 4 cm^2. Samantha's table and diagram are shown

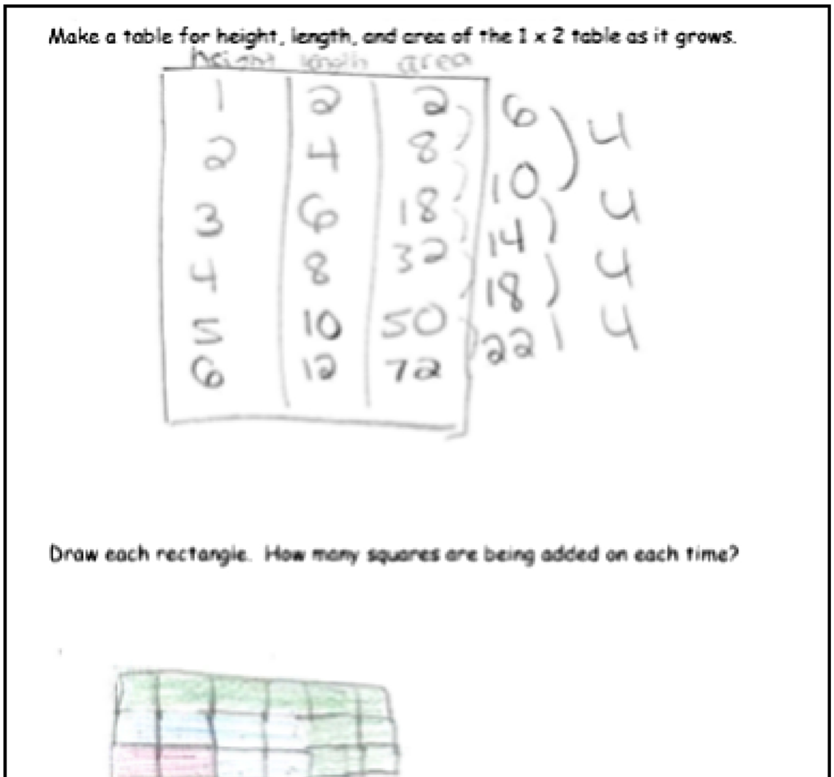


Fig. 11. Samantha's work on the 1 × 2 growing rectangle task.

in Fig. 11.

The TR pressed, “So, any idea why it’s 4, instead of 2 or 5 or something?” Jim explained, “When the dimensions of the box change, then the you have your rate of rate of growth of the rate of growth are different.” The TR continued to press for students to connect their reasoning to the quantities height and length:

TR: What does it have to do with the dimensions of the rectangle? In other words, what's the 4 got to do with the dimensions of my original rectangle which I believe was uh…1 to 2? Bianca?

Bianca: “Well, it like…it has to do with the ratio, but I really can't explain it.”

In this exchange, Jim explained that the constantly-changing rate of change depended on the dimensions of the rectangle, but he was not exactly sure what that relationship was. Bianca built on Jim’s idea to conjecture the it was related to the ratio (of the height to length), but could not articulate how exactly the ratio affected the difference in the rate of growth of area. In a later exchange, Jim expressed, “So your rate of growth (for the area) can change no matter what.”

The students understood that the quantities height and length affected the constantly-changing rate of change of the area, but could not yet identify precisely how those quantities, or changes in them, affected the change in area. We take this as evidence of *WoU4.1 Recognition that Change in One Quantity Determines Change in the Other*. The instructional goals were focused on helping students recognize and name relationships among linked, changing quantities. Tasks were designed to support students to see a dependency relationship between changing quantities with explicit attention to identifying those quantities in both pictures of the growing rectangle and tables of values.

4.3.2. WoU4.2 identification of how change in one quantity determines change in the other

Eventually the students were able to transition to not only understanding *that* changes in one quantity determine changes in another, but they could also identify *how* this occurred, particularly in terms of how changes in height and/or length affected the constantly-changing rate of change of area. For instance, for the Four Tables task (see Fig. 12), Tai found a way to relate the constantly-changing rate of change of the area with respect to the changes in both the height and the length. Tai said: “Take the [difference in the change] of the area, and you divide by the difference in the length…and also divided by, the difference in height… and, always equals 2.” Tai identified a dependency relation between ΔH, ΔL, and ΔΔA; he verbally expressed the relationship $\left[\frac{\frac{\Delta\Delta A}{\Delta L}}{\Delta H}\right.$ = 2.]. Another student, Bianca, noticed that ΔΔA could be expressed in terms of change in height values: “So it’s basically like 3 [times difference in height] squared times 2.” Tai expressed this finding as the equation: 3•[difference in height]^2 = [constantly-changing difference in area].” We call this *WoU4.2 Identification of How Change in One Quantity Determines Change in the Other*. In this WoU, students articulated how changes in a quantity such as height or length determined changes in the change in Area, for example.

We observed the *WoU4.2* to emerge in the context of deliberate norms, teacher moves, and task design principles. The students approached the Four Tables task by making changes in linked quantities explicit in the table representation, which had become an established norm. Prior to the Four Tables task, the class had investigated other tasks that encouraged prediction of the difference in the rate of growth. In one task, students predicted the constantly-changing rate of change in a rectangle’s area given only the ratio of its height to length (recall Fig. 9b, for example). These tasks led the class to explore several student conjectures, each of which demanded a quantitatively-based justification. We surmise that this repeated reasoning of making and testing predictions cultivated a norm that encouraged the expression of dependency relationships. Finally, the tasks were designed to: (a) investigate a quadratic growth situation given an initial independent quantity greater than 1, (b) investigate a collection of tables with varied changes in the independent quantity, and (c) elicit conjectures about generalized relationships among linked quantities for a quadratic growth context. The sequencing of tasks of the same type (such as the prediction tasks and the Four Tables task) encouraged the students to

Here are 4 tables for the height versus the **area** of the exact same rectangle that is growing in proportion

Height	Area
1	3
2	12
3	27
4	48
5	75

Height	Area
2	12
4	48
6	108
8	192
10	300

Height	Area
5	75
10	300
15	675
20	1,200
25	1,875

Height	Area
10	300
20	1,200
30	2,700
40	4,800
50	7,500

In each case, what does the formula for area have to do with the constantly-changing rate of change of the area?

Fig. 12. The Four Tables Task.

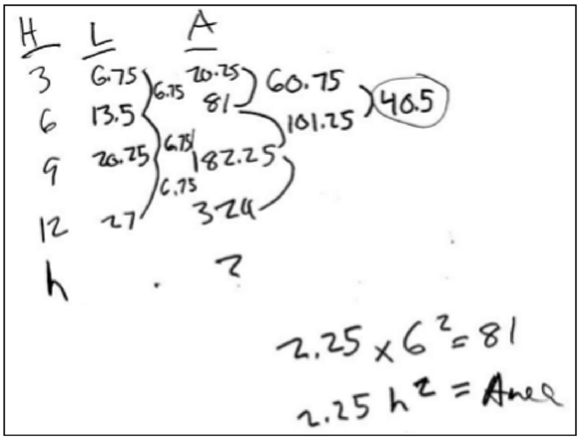


Fig. 13. Jim's table and equation relating height and area of a growing rectangle.

engage in repeated reasoning to test and/or formalize conjectures about generalized relationships. We call attention to these generalizing activities because ultimately, this kind of generalizing also helped the students develop quadratic equations, as we elaborate next in the final WoU.

4.3.3. WoU4.3 translations of dependency relations of change to correspondence rules

The *WoU4.3 Translations of Dependency Relations of Change to Correspondence Rules* emerged when the students leveraged their coordination thinking to create and make sense of correspondence rules. At this point in the TE, many instructional supports had become normative. The students repeatedly engaged in reasoning that led them to identify and quantify dependency relations of change. Further, the TR implemented carefully sequenced sets of tasks aimed at prompting students to link their understandings of dependency relations of change to a symbolic rule. For example, the TR set up a task asking students to write an equation to: (a) find the area of the rectangle when the height is h, and (b) relate the equation to the constantly-changing rate of change of area with respect to height. Consider Jim's work on this task, shown in Fig. 13.

The students hypothesized that the difference in the rectangle's length divided by the difference in the height was the coefficient $a = \frac{\textit{difference in length}}{\textit{difference in height}}$ for the symbolic rule $Area = ah^2$. The students explained:

Jim: 6.75 divided by three equals 2.25, so if I did 2.25 times six squared equals 81. It works. 2.25 times h^2 equals area.

TR: So what works?

Daeshim: Um, dividing by um, dividing by how many are going up by each time. So, like the, what's in the length, divided, divided by how much [inaudible] by height is equal to.[4]

In this exchange, the students expressed the relation between changes in two quantities as an algebraic rule. The task was designed to have students give a far prediction for the area and to elicit a generalization of the relationship between the height and the area. The students were accustomed to linking quantities as relationships of explicit coordinated change, and making and testing predictions about generalized relationships. The TR made these normative practices explicit by asking the students to test their hypothesis and to state a generalized rule. When students stated their rules, the TR then pressed for quantitatively-based justifications: "So, it's this difference in the length divided by three, and why divided by three?" to which the students gave a quantitatively-based answer "Because that's what you're going up by each time." A summary of the WoU within the *WoT4 Dependency Relations of Change* is given in Table 5.

4.4. Transition IV: establishing correspondence

In the final transition to *WoT5 Correspondence*, we found one main WoU: *Correspondence Relations Between Independent and Dependent Quantities* (*WoU5.1a, WoU5.1b*). We identified nine instructional supports: *Teacher Moves* (1) Ask students to create an algebraic rule that relates an independent and dependent quantity; (2) Press for quantitatively-based justifications; (3) Ask students to generalize a mathematical relationship they identified; *Task Design Features* (4) Investigate a quadratic growth situation where change in independent quantity is greater than 1; (5) Make far predictions of independent and dependent quantities for a given quadratic growth pattern; (6) Elicit generalizations of relationships between independent and dependent quantities using variables; (7) Sequence tasks of the same type to allow students to test conjectures about generalized relationships; and *Norms* (8) Be explicit about the magnitude of change for each of the relevant quantities; and (9) Make and test predictions about generalized relationships

[4] In reading Jim's work in Fig. 13, note that the generalization that $a = \frac{\textit{difference in length}}{\textit{difference in height}}$ was not written down, but rather expressed verbally by both Jim and Daeshim.

Table 5
WoT4 Dependency Relations of Change, Related WoU, and Instructional Supports.

WoT4 Dependency Relations of Change (conceiving of the change in one quantity as dependent on other quantities or change in other quantities)			
WoU	Definition	Data Example	Instructional Support
WoU4.1 Recognition that Change in One Quantity Determines Change in the Other	The magnitude of the change of one quantity, such as height, determines the amount of change in another quantity, such as area (or the difference in the rate of growth of area). Student understands that there is a dependency relation *without* determining what that relation is.	Jim: "When the dimensions of the box change, then the you have your rate of rate of growth of the rate of growth are different."	*Norms:* ● Attend to linked quantities as relationships of explicit coordinated change; ● Create tables and drawings to express relationships in quadratic growth contexts; ● Multiple student responses are shared for a single question or task; and ● Students respond to and build on each other's ideas. *Task Design Feature:* ● Quadratic growth task allows for multiple different ways of interpreting and explaining one's reasoning. *Teacher Move:* ● Press for quantitatively-based justifications.
WoU4.2 Identification of How Change in One Quantity Determines Change in the Other	The student quantifies the relation between change in one quantity and the difference in the rate of growth of change in the other quantity (area).	Tai: "(Referring to a table of values coordinating successive height (H), length (L), and area (A) values for a growing rectangle) [constantly-changing difference of change] divided by [difference in length] equals…and, divided by [difference in height] equals 2. [Here he verbally expressed the relationship $\left[\frac{\frac{\Delta\Delta A}{\Delta L}}{\Delta H}\right] = 2$.]"	*Task Design Features*: ● Investigate a quadratic growth situation given an initial independent quantity (x) greater than 1; ● Investigate a collection of tables for quadratic growth where the change in the independent quantity (Δx) is greater than 1 and varied across the set; ● Elicit conjectures about generalized relationships among linked quantities for a quadratic growth context; and ● Sequence tasks of the same type to allow students to engage in repeated reasoning to test and/or formalize conjectures about generalized relationships. *Teacher Move*: ● Press for quantitatively-based justifications. *Norm:* ● Make and test predictions about generalized relationships among linked quantities; and ● Attend to linked quantities as relationships of explicit coordinated change.
WoU4.3 Translation of Dependency Relations of Change to Correspondence Rules	Expressing the relation between changes in two quantities in an algebraic rule, such as $y = ax^2$. Student understands the rule in terms of a relation of change.	Jim: "look…[the constantly-changing difference of change] divided by 2 is [the difference in the length]" Bianca: "[the constantly changing rate of change] over 2 equals [the difference in the length]." TR: "What does [the difference in length] have to do with area?" Bianca: "Distance, length. Distance times length." Jim: "[difference in length] is, um, this. The 3h squared."	*Task Design Feature*: ● Sequence tasks of the same type to allow students to test conjectures about generalized relationships. ● Investigate a quadratic growth situation where change in independent quantity (Δx) is greater than 1; and ● Elicit conjectures about generalized relationships among linked quantities for a quadratic growth context. *Norm*: ● Attend to linked quantities as relationships of explicit coordinated change; and ● Make and test predictions about generalized relationships among linked quantities. *Teacher Moves:* ● Press for quantitatively-based justifications; and ● Ask students to generalize a mathematical relationship they identified.

among linked quantities.

4.4.1. WoU5.1 correspondence relations between independent and dependent quantities

For each correspondence rule that the students wrote, the TR asked them to explain what each variable meant. This press for quantitatively based explanations supported the students to write rules that were grounded in quantitative meaning. For example, Bianca developed a relationship between the side of a growing square and the area of a growing square as "$a = s^2$" or "the area = the side measurement squared". The TR prompted Bianca to further connect this rule to the relevant quantities.

TR: Area equals side squared [writes on the board]. So can you give me an example of that. Like, for instance, this point (4,16)—

Bianca: The 4 is the…4 which is the height and the 4 which is the width [gestures to the height and width of the projected square].

TR: Ok. So the 4 is the height and the length.

Bianca: Yeah.

TR: Where's the 16 in the picture?

Bianca: It's the number of squares inside.

In this example, Bianca articulated both a general correspondence rule relating independent and dependent quantities, $a = s^2$ and a specific instance, of how 4 cm and 4 cm are the side lengths and 16 cm^2 is the area, which is why (4,16) works for the rule $4^2 = 16$. In related tasks, the TR asked far prediction questions for very large height values. For instance, given a table of height / area values for a 2 cm by 9 cm rectangle with height values ranging from 2 cm to 8 cm, the TR asked the students to determine the area for a height of 82:

TR: Yeah, so you found out the area when the height's 82. You found it to be 30,258. What if height's n?

Jim: Oh, I think I got it: n times 4.5 times n equals area. Simple. I think.

In Far Prediction tasks, the TR asked students to generalize mathematical relationships by giving prompts such as "what would [the area] be if the height is just h?" Sometimes, those prompts were also accompanied by a prompt to write an algebraic rule. The normative practices of being explicit about the amount of change for each of the relevant quantities, as well as making and testing predictions about generalized relationships, remained instructional supports in *WoT5 Correspondence*. The students' engagement in repeated reasoning through a sequence of tasks of the same type was also evidence of an instructional support. A summary of these examples with related definitions is given in Table 6.

5. Discussion and conclusion

In this research we sought to better understand how one can engender and model conceptual change in students' WoT and WoU quadratic growth. The students who participated in this study went through four major transitions between five WoT: *Variation*, *Early Coordinated Change*, *Explicitly Quantified Coordinated Change*, *Dependency Relations of Change*, and *Correspondence*. These transitions did not occur spontaneously. We deliberately use the word transition to convey a broader interaction modeling changes in both students' WoU and WoT, and changes in the instructional supports and context vis-à-vis teacher moves, norms co-developed among the TR and the students, and task design features.

Our findings suggest that meaningful learning of quadratic growth is conceptually challenging for middle-grades students, but possible to attain with tailored instructional supports. A rate of change approach to quadratic growth, coupled with design principles grounded in DNR-based instruction, supported the teaching and learning of quantitative reasoning, representational fluency, and generalization in a dynamic, visualizable context. The learning trajectory we developed offers insight into a model of students' mathematics, and how instruction might foster a dynamic understanding of function, which is critical for success in higher-level mathematics.

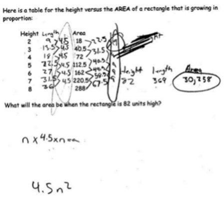
5.1. Instructional supports for students' WoT and WoU quadratic growth

Although learning and teaching function in conceptually oriented ways remains central in secondary mathematics education, there remain well-documented challenges for both students and teachers (e.g., Wilkie, 2019). In light of the promise of instructional tasks with an emphasis on a rate of change perspective, there remains a need for additional evidence that links students' meaningful learning with effective instructional supports. This study addressed this gap by elaborating types of instructional supports for students' WoT and WoU about quadratic growth. From the design phase of this research, the instructional supports, particularly the tasks, were deliberately engineered to foster and, ultimately, require students to explicitly identify the manner in which two quantities changed together. Our instructional supports were grounded in a theoretical orientation toward the importance of fostering students' quantitative reasoning, and a mathematical goal of supporting students' understanding of quadratic growth as a relationship between co-varying quantities in which one quantity changes at a constantly-changing rate of change with respect to the other. Our conceptually oriented mathematical goals remained central to our design, yet did not overshadow the importance of building models of students' mathematics as the TR and students interacted around a purposeful instructional task sequence.

These design decisions, and a DNR-based approach to instruction, certainly background our findings. However, our categorization

Table 6
WoT5 Correspondence, Related WoU, and Instructional Supports.

WoT5 Correspondence (Conceives of direct correspondence relation between independent quantity and dependent quantity)

WoU	Definition	Data Example	Instructional Support
WoU5.1a Specific Correspondence Between Independent and Dependent Quantities	Student understands a correspondence relation between independent quantity x (height or length) and dependent quantities y (area) for a specific value or values of a relationship.	TR: "Area equals side squared [writes on the board]. So can you give me an example of that. Like, for instance, this point (4,16)—" Bianca: "The 4 is the…4 which is the height and the 4 which is the width [gestures to the height and width of the projected square]." TR: "Ok. So the 4 is the height and the length." Bianca: "Yeah." TR: "Where's the 16 in the picture?" Bianca: "It's the number of squares inside."	*Teacher Moves*: ● Ask students to create an algebraic rule that relates an independent and dependent quantity; ● Press for quantitatively-based justifications; and ● Ask students to generalize a mathematical relationship they identified.
WoU5.1b. Generalized Correspondence Rule	Student understands the function rule as a generalized instance of a mapping between independent and dependent quantities. Student expresses a correspondence rule as a direct relationship between an independent quantity (height or length) and dependent quantity (area). This may be expressed in words, in a diagram, or in algebraic symbols (e.g., $y = mx^2$ or $A = ah^2$).	TR: "Yeah, so you found out the area when the height's 82. You found it to be 30,258. What if height's n?" Jim: "Oh, I think I got it. n times 4.5 times n equals area. Simple. I think." 	*Task Design Features*: ● Investigate a quadratic growth situation where change in independent quantity (Δx) is greater than 1; ● Make far predictions of independent and dependent quantities for a given quadratic growth pattern; ● Elicit generalizations of relationships between independent and dependent quantities using variables; and ● Sequence tasks of the same type to allow students to test conjectures about generalized relationships. *Norms*: ● Be explicit about the magnitude of change for each of the relevant quantities; and ● Make and test predictions about generalized relationships among linked quantities.

of instructional supports into teacher moves, norms, and task design features was emergent. We found the teacher moves and norms to explicitly foster a discourse community that focused on quantitative reasoning, representational fluency, and generalization. We found the task design features to fall along related categories of repeated reasoning, far prediction and generalization, and doing mathematics.

The set of instructional supports introduced in this paper offers a new framework for elaborating some of the mechanisms that can support students' conceptual change. We found that the tasks and task design features, teacher moves, and norms interactively worked together to support transitions in students' WoT and WoU. An example of the integrated nature of instructional supports was given in Transition II; the teacher moves and task design features did not work in isolation, but rather in concert with carefully developed norms such as being explicit about the magnitude of change in linked quantities. For instance, introducing tables incremented by units less than 1 cm in height encouraged the development of a partial-unit coordination of changes in height values with changes in area values. These tables, however, would likely not have been effective in supporting this coordination had the students not already become accustomed to explicitly identifying the change in height. This was a norm that established slowly over the course of multiple task sequences, and it was only once this norm was firmly in place that the students were able to begin explicitly coordinating changes for increments less than 1. Similarly, a number of the associated teacher moves, such as pressing for quantitatively-based justifications, were made possible and effective in the context of tasks situated in a quantitatively-rich situation and norms encouraging students to build on one another's ideas. It is the interaction between task design, teacher moves, and norms that enable transitions in students' WoT and WoU; tasks alone cannot carry the full responsibility for enacting conceptual change.

5.2. Quantitative reasoning about functions and DNR

We found that the students' WoT about quadratic growth (*WoT1* to *WoT5*) developed in reflexive relation with more specific WoU within the particular content of quadratic growth. Recall the left-most column of Table 1, and the left-hand side of Fig. 4. This research extends current characterizations of students' conceptual learning of quadratic function (e.g., Ellis, 2011a, 2011b; Wilkie, 2019), and taxonomies for conceptual learning goals (e.g., Lobato et al., 2012) by elaborating the nature of transitions among

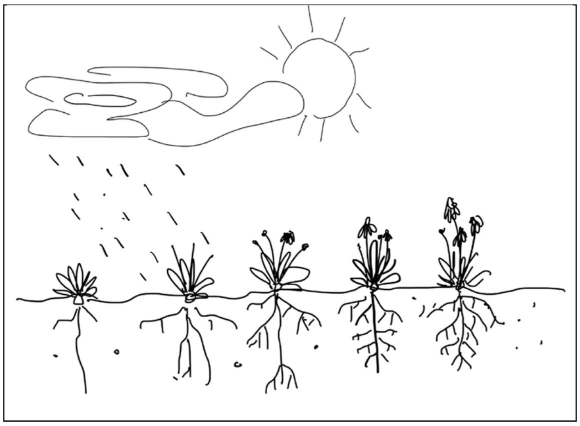


Fig. 14. A visual metaphor for a learning trajectory of change in models of students' mathematics (flower) fostered by a system of interacting instructional supports (sun, rain, soil).

qualitatively distinct ways of thinking. In particular, this research contributes to a growing body of work on learning trajectories that explicitly attends to the duality of students' broader ways of thinking about mathematics content and specific ways of understanding mathematical ideas (Empson, 2011).

This research also provides nuance into the kinds of WoT that are involved in reasoning about quadratic function, expanding on Harel's (2013) framing of the kinds of broader conceptual tools students bring to bear in a specific learning context. *Early Coordinated Change* and *Dependency Relations of Change*, for example, extend current framings of quantitative reasoning about functions, and quadratic functions in particular. In common characterizations of students' quantitative reasoning about functions, much attention is given to explicit coordinated change (cf. *WoT3*; see also Confrey & Smith, 1994) or covariation (Castillo-Garsow, 2013; Saldanha & Thompson, 1998), and correspondence (cf. *WoT5*) reasoning about functions (see also Smith, 2003).

5.3. Visualizing learning trajectories

In writing about the quadratic growth learning trajectory as a series of transitions between WoT and instructional supports we do not intend to convey a linear progression of ideas, nor do we intend to convey a neat structuring of developmental stages that occur in sequence. To illustrate an example of how learning is not linear, *WoT5 Correspondence* tended to co-occur with *WoT4 Dependency Relations of Change* (this co-occurrence is illustrated in Fig. 4). We choose to introduce *WoT5* last in Transition IV because it tended to occur late in students' thinking, and it was instructionally last in the sequence. Different students will inevitably traverse a learning trajectory in different ways. Our construction of a learning trajectory in this paper relied on our best model from the data as guided by our theoretical orientation. We do not claim this is the only learning trajectory for quadratic growth.

We find the metaphor of growing flowers in their appropriate habitat to be a useful tool for thinking about the interrelationships among our theory-driven design, characterizations of students' WoT and WoU, and articulation of instructional supports as a set of teacher moves, task design features, and norms. In Fig. 14 we introduce a Visual Metaphor depicting transitions in both the growth of the plant (change in the mathematics of students), as well as an interacting system of soil, water, and sun as a nourishing environment engendering growth (instructional supports). This metaphor affords flexibility in conceptualizing both the interaction of a growing flower with the environment, as well as the system of environmental interactions. For learning trajectories, this visual metaphor makes these two features salient: (a) how the child interacts with the learning environment, and (b) how the tasks and task design features, teacher moves, and norms interactively work together to support children's learning.

Often learning trajectories positing models of students' mathematics are taken to be "just a bunch of flowers" without due attention to the connectedness of how flowers grow in response to the environment nurturing them. With this visual metaphor, just as how different ecological environments nurture different kinds of plant growth, we can imagine how different kinds of learning environments and instruction might support qualitative differences in children's learning.[5] The learning trajectory developed in this study contributes evidence of how change in students' WoU and WoT can be engendered by theory-driven instructional supports. More broadly, the paper contributes an example of how learning trajectories research can attend to both the nature of students' learning and the nature of supports for learning (beyond a sequence of tasks).

[5] We thank an anonymous reviewer for pointing out that certain plants are not suited to grow in certain climates. If a plant does not grow in a certain environment, the problem is not with the plant. Likewise, if a child does not learn in a particular instructional setting, the problem is not with the child. We do not intend for the visual metaphor of a learning trajectory to convey a deficit orientation toward children. Rather, the metaphor highlights the importance of both understanding and supporting the richness of students' mathematics as contextualized in learning environments.

5.4. Concluding remarks

Theory development that links theory of learning and theory of instruction is needed (Simon et al., 2010). We see this study as an example of research that deliberately attends to both learning and teaching to explain change in students' WoU and WoT about mathematics. Our learning trajectory is a networking of theories of learning and theories of instruction that are grounded in empirical observation of students' mathematical activity. It is a nuanced model linking conceptual learning goals, models of students' WoT and WoU, and instructional supports that can engender students' transitions from one WoT to another. True to developmental research (Brown, 1992; Gravemeijer, 1994) our theoretical orientation guided both the engineering of the learning and teaching context and our analysis of data. For example, the TR was active in hypothesizing students' activity, and chose to enact instructional supports that extended students' reasoning, effectively engendering a shift in students' WoT. We see great potential in expanding research and practice around the use and creation of learning trajectories that link these inter-related dimensions of teaching and learning. Learning trajectories research can help propel the field by unifying research programs that seek to understand what students' conceptual learning of mathematical ideas and WoT can look like, and how instruction can support that learning.

CRediT authorship contribution statement

Nicole L. Fonger: Conceptualization, Formal analysis, Visualization, Writing - original draft, Writing - review & editing. **Amy B. Ellis:** Conceptualization, Formal analysis, Funding acquisition, Writing - original draft, Writing - review & editing. **Muhammed F. Dogan:** Data curation, Formal analysis, Writing - review & editing.

Appendix A. Supplementary data

Supplementary material related to this article can be found, in the online version, at doi:https://doi.org/10.1016/j.jmathb.2020.100795.

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Empirical Re-Conceptualization: Bridging from Empirical Patterns to Insight and Understanding

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Identifying patterns is an important part of mathematical investigation, but many students struggle to explain or justify their pattern-based generalizations or conjectures. These findings have led some researchers to argue for a de-emphasis on pattern-based activities, but others argue that empirical investigation can support the discovery of insight into a problem's structure. We introduce a phenomenon we call empirical re-conceptualization, in which learners identify a conjecture based on an empirical pattern, and then re-interpret that conjecture from a structural perspective. We elaborate this construct by drawing on interview data from undergraduate calculus students and research mathematicians, providing a representative example of empirical re-conceptualization from each participant group. Our findings indicate that developing empirical results can foster subsequent insights, which can in turn lead to justification and proof.

Keywords: Generalization, Patterns, Conjecturing, Justification

Introduction and Motivation

Identifying patterns is a fundamental aspect of mathematical activity, with curricular materials and instructional techniques geared towards supporting students' abilities to leverage empirically-based generalizations and conjectures. However, forming pattern-based conjectures is not sufficient; it is also important for students to understand and justify the patterns they develop. A robust body of research reveals a common phenomenon whereby students are able to leverage patterns in order to develop conjectures, but then struggle to understand, explain, or justify their results (e.g., Čadež & Kolar, 2014; Mason, 1996; Pytlak, 2015; Zazkis & Liljedah, 2002). Indeed, these findings have led some researchers to argue for a de-emphasis on pattern-based activity, dismissing it as unsophisticated (e.g., Carraher et al., 2008; Mhlolo, 2016).

In contrast, we have observed a phenomenon we call *empirical re-conceptualization*, in which participants identify a pattern based on empirical evidence alone to form a conjecture, and then re-interpret their conjecture from a structural perspective. In this paper, we address the following research questions: (a) What characterizes students' and mathematicians' abilities to leverage empirical patterns to develop mathematical insights? (b) What are the potential affordances of engaging in empirical patterning activity? We describe and elaborate this construct across two participant populations, mathematicians and undergraduate students. In this manner, we highlight empirical re-conceptualization as a phenomenon that marks productive mathematical activity at multiple levels and populations. Our findings indicate that developing results from empirical patterns, even those that are poorly understood or unjustified in the moment, can serve as a launching point for subsequent insights, including verification, justification, and proof.

Literature Review and Theoretical Perspectives

There is ample evidence that students are adept at identifying and developing mathematical patterns (e.g., Blanton & Kaput, 2002; Pytlak, 2015; Rivera & Becker, 2008). However, the patterns students identify may not always be those that are mathematically useful. As Carraher et al. (2008) noted, a pattern is not a well-defined concept in mathematics, and there is little agreement on what constitutes a pattern, much less its properties and operations. Students who do identify patterns can then experience difficulties in shifting to algebraic thinking (e.g., Čadež & Kolar, 2015, Moss, Beatty, McNab, & Eisenband, 2006; Mason, 1996). Further, both secondary and undergraduate students who work with patterns struggle to justify them (Hargreaves et al., 1998; Zazkis & Liljedah, 2002). An emphasis on empirical patterning without meaning can promote the learning of routine procedures without understanding (Fou-Lai Lin et al., 2004; MacGregor & Stacey, 1995), or the generalization of a relationship divorced from the structure that produced it (Küchemann, 2010). Further, students' challenges with extending pattern generalization to meaningful learning has been shown to contribute to difficulty in multiple domains, including functions (Ellis & Grinstead, 2008; Zaslavsky, 1997), geometric relationships (Vlahović-Štetić, Pavlin-Bernardić, & Rajter, 2010), and combinatorics (Kavousian, 2008; Lockwood & Reed, 2016), among others.

Despite these drawbacks, researchers also point out the affordances of empirical investigation and pattern development. The activity of developing empirically-based conjectures can support the discovery of insight into a problem's underlying structure, which can, in turn, foster proof construction (Tall et al., 2011; de Villiers, 2010). The degree to which pattern generalization is an effective mode of proof development is an unresolved question, but there is evidence that students can engage in a dynamic interplay between empirical patterning and deductive argumentation (e.g., Schoenfeld, 1986). Similarly, research mathematicians regularly engage in experimentation and deduction as complementary activities (de Villiers, 2010). It may be that students become stuck in a focus on empirical relationships divorced from structure because they lack sufficient experience with this way of thinking. Küchemann (2010) found that with practice, students could learn to glean structure from patterns. Similarly, Tall et al. (2011) argued that attention to the similarities and differences in empirical patterns could support the development of mathematical thinking and proof. These works offer a precedent for positioning empirical patterning as a bridge to insight and deduction.

Structural reasoning. Harel and Soto (2017) introduced five major categories of structural reasoning: (a) pattern generalization, (b) reduction of an unfamiliar structure into a familiar one, (c) recognizing and operating with structure in thought, (d) epistemological justification, and (e) reasoning in terms of general structures. The first category, pattern generalization, further distinguishes between two types of generalizing: *Result pattern generalization*, and *process pattern generalization* (Harel, 2001). Result pattern generalization (RPG) is a way of thinking in which one attends solely to regularities in the result. The example Harel gives is observing that 2 is an upper bound for the sequence $\sqrt{2}$,$\sqrt{2 + \sqrt{2}}$, $\sqrt{2 + \sqrt{2 + \sqrt{2}}}$, … because the value checks for the first several terms. RPG is typically the type of pattern generalization observed in studies in which students then struggle to shift from recursive to explicit relationships or justify their patterns (e.g., Čadež & Kolar, 2015; Schliemann, Carraher, & Brizuela, 2007). When we refer to the identification of a pattern based on empirical evidence, we are referring to RPG. In contrast, process pattern generalization (PPG) entails attending to regularity in the process, even if that attention may first be initiated by noticing a regularity in the result (Harel, 2001). To extend the above example, Harel discussed how one might engage in PPG to determine that there is an invariant relationship between any two consecutive terms of the sequence, $a_{n+1} = \sqrt{a_n + 2}$, and therefore reason that all of the terms of the sequence are bounded by 2 because $\sqrt{2} < 2$.

We define empirical re-conceptualization as the process of re-interpreting a generalization or conjecture from a pattern (identified by RPG) from a structural perspective. By structural perspective, we mean the five major categories of structural reasoning, with the exception of the RPG sub-category of pattern generalization. Thus, a student could reason about the regularity in process from one term to the next of a sequence (PPG). One could also reduce unfamiliar structures into familiar ones, either by constructing new structures or by forming conceptual entities. One may also carry out structural operations in thought without performing calculations, reason in terms of general structures (either by reasoning with conceptual entities or reasoning with operations on conceptual entities), or engage in epistemological justification. In short, re-interpreting a generalization or conjecture from a structural perspective entails the ability to recognize, explore, and reason with general structures.

Figurative and operative thought. The other construct we draw upon to characterize the phenomenon of empirical re-conceptualization addresses a distinction in mental activity (Piaget, 1976, 2001; Steffe, 1991; Thompson, 1985). When engaged in figurative activity, one attends to similarity in perceptual or sensorimotor characteristics. In contrast, operative mental activity entails attending to similarity in structure or function through the coordination and transformation of mental operations. For instance, a student could associate the sine curve with circular motion through conceiving both as representing an invariant relationship of co-varying quantities (an operative association), or through conceiving both as smooth because the motion is perceived as continuous (a figurative association) (Moore et al., 2019). A shift from RPG to PPG is often accompanied by a shift from figurative to operative mental activity, and we consider operative activity to be a hallmark of the ability to reason structurally.

Methods

We drew on interview data from two participant sources, mathematicians and undergraduate students, both stemming from larger projects investigating participants' use of examples to generalize, conjecture, and prove.

Mathematician data. The mathematician data consisted of two sets of hour-long interviews. Thirteen mathematicians participated in Interview 1, and 10 continued for Interview 2. The participants included 7 professors, 3 postdoctoral researchers, and 3 lecturers. Twelve participants hold a Ph.D. in mathematics, and one holds a Ph.D. in computer science. There were 8 men and 5 women. Each interview presented two novel mathematics tasks, which were chosen to be accessible (i.e., they did not require specialized content knowledge) but not trivial (i.e., a solution was not immediately available). For the purposes of this report, we focus on the *Interesting Numbers Task* (Andreescu, Andrica, & Feng, 2007), which we phrased as follows: "Most positive integers can be expressed with the sum of two or more consecutive integers. For example, 24 = 7 + 8 + 9, and 51 = 25 + 26. A positive integer that cannot be expressed as a sum of two or more consecutive positive integers is therefore interesting. What are all the interesting numbers?" Below we highlight an exemplar from one mathematician's work on the interesting numbers task, the work of Dr. Fisher.

Undergraduate data. The undergraduate data consisted of a set of hour-long individual interviews with 10 undergraduate male calculus students. The students solved a set of tasks designed to engender generalizing activity (and ultimately to generalize the binomial theorem). For this paper, we report on the *Passwords Task*, which asked students for the number of passwords of length 3, 4, 5, and eventually length n, which consisted of the characters A or B, where repetition was allowed. We asked the students to create tables to organize passwords with a certain number of As, and we also had them reason about the total number of passwords of a given length. Below we focus on one student's work, Raoul, and his reasoning about the total number of length n AB passwords.

Analysis. All interviews were videoed and transcribed, and were also recorded with a Livescribe pen, which yields both an audio record of the interview and a pdf document of the participant's written work. We used gender-preserving pseudonyms for all participants. Using the constant-comparative method (Strauss & Corbin, 1990), we analyzed the interview data in order to identify the participants' generalizations and conjectures and to characterize the mental activity that fostered them. For the first round of analysis we drew on Ellis et al.'s (2017) *Relating-Forming-Extending* Framework for generalizing, which yielded the emergent category of empirical reconceptualization. In subsequent rounds we further examined the participants' talk, gestures, and task responses to further refine the characteristics of empirical reconceptualization.

Results

In order to characterize the participants' abilities to leverage empirical patterns to develop mathematical insights, we present two exemplar cases of empirical re-conceptualization, one from each participant group. In both cases, the participants began with an empirically-based conjecture that they did not understand and could not justify. That initial conjecture, however, then served as a launch point to engage in empirical reconceptualization.

Mathematician Case: Dr. Fisher

Dr. Fisher is a mathematics professor at a large public university. She initially approached the Interesting Numbers Task algebraically by expressing the non-interesting numbers as $m + m + 1 + \cdots + m + k - 1$. After trying to simplify that expression, she switched to a different approach, in which she listed all of the interesting numbers by ruling out those that were non-interesting. She checked the first numbers 1 to 7 in her head, noting that 3 = 1+2, 5 = 2+3, 6 = 1+2+3, and 7 = 3+4, so only 1, 2, and 4 were interesting. Then she realized she could list all of the non-interesting numbers as sums in an organized table (Figure 1):

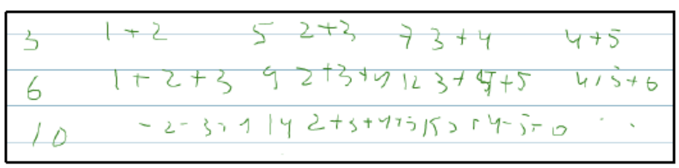

Figure 1. Dr. Fisher's initial table.

Using her table, Dr. Fisher was able to rule out every number up to 24 other than 1, 2, 4, 8, 16, 19, and 23. She recognized the pattern in the first five terms as the powers of 2 and (correctly) conjectured that the interesting numbers were the powers of 2. She made this conjecture even though she had not yet been able to rule out 19 and 23, noting that, "19 and 23 are sort of bothering me from the power of 2 that is showing up in the pattern." Dr. Fisher's conjecture emerged from RPG and figurative activity, in which she noticed a familiar pattern in the (incomplete) data. However, at this time she had no sense of why it should continue.

To determine why the powers of 2 were interesting, Dr. Fisher rewrote her table (Figure 2). She then suddenly noticed a structural pattern in the sums, observing that the sums in the second column could be obtained from the sums in the first column moving diagonally by removing the "+1". Similarly, she saw that the sums in the third column could be obtained from the sums in the second column moving diagonally by removing the "+2". In other words, when moving diagonally up the table, 1+2+3+4 becomes 2+3+4, which becomes 3+4.

1	2	3
1+2	2+3	3+4
1+2+3	2+3+4	3+4+5
1+2+3+4	2+3+4+5	3+4+5+6
1+2+3+4+5	2+3+4+5+6	3+4+5+6+7

Figure 2. Dr. Fisher's refined table attending to sums.

> *Dr. Fisher:* Oh. I see. I see what's happening. […] So, what I see here is that they differ by 1, right? And then they differ by 2. Right? So, I have these numbers [in the first column] and then automatically get these minus 1. […] And then I automatically get those numbers minus 1 and then I get them minus 2 also. And then I get the minus 3 et cetera. So, this actually should tell me most of them by just subtracting.

This insight led Dr. Fisher to formalize the non-interesting numbers as $\binom{n}{2} - 1 - 2 - \cdots - k$, which she simplified to $\binom{n}{2} - \binom{k}{2}$. However, she still did not understand why the powers of 2 did not have that form. To try to answer that question, she returned again to the table and rewrote a third version, this time listing the values of the sums rather than the sums themselves (Figure 3). She evaluated the first row of sums (1+2, 2+3, 3+4, …), then filled in the rest of the table using the diagonal relationship she observed earlier.

3	5	7	9
6	9	12	15
10	14	18	22
15	20	25	30

Figure 3: Dr. Fisher's third refined table.

During this activity, Dr. Fisher noticed a pattern in the second row, "Now what I'm getting here is that these are multiples of 3. Yes, I got it!" To prove this observation, she represented an arbitrary element of the first row by $2k+1$, then used the diagonal relationship to justify that the next element down the diagonal would be $(2k-1) + (k-2) = 3(k-1)$. She gave a similar proof for the multiples of 5 appearing in the 4th row, then realized that a generalized version of the argument would prove that every multiple of $2k+1$ (above a certain point) will appear in the $2k$ row. Her argument proved that every multiple of an odd number is non-interesting, so every interesting number must have no odd factors (i.e. be a power of 2).

Dr. Fisher initially engaged in RPG, generating a conjecture based solely on the numerical pattern she observed in her list of interesting numbers. Once she had the conjecture, however, she shifted from figurative to operative activity, attending to how the sums changed as the number of summands and starting value changed within her table. Through her analysis of the structural relationships in the data, Dr. Fisher was able to engage in multiple rounds of PPG, forming generalizations that led to a partial proof of her conjecture.

Undergraduate Case: Raoul

Raoul was an undergraduate calculus student at a large public university. In his work on the Passwords Task, Raoul leveraged an empirical pattern to develop a conjecture that for an n-character password made of As and Bs, there would be 2^n total passwords. In the excerpt below, Raoul explained that he saw the pattern based his prior work determining 3-character, 4-character, and 5-character passwords. However, he did not understand why the total number of passwords was 2^n:

Raoul: Over here I get, for 3 characters, I get 8 numbers. Four characters, 8 times 2, 16. Five characters, 16 times 2. Oh! 2 power n.
Interviewer: Okay. Why did you think that?
Raoul: Well I guess, I just started seeing the pattern. I mean 8 is the 2 cubed. I knew that, and I knew that 5 is…no sorry, 32 is 2 power 5. I knew that too, and here is the same, 2 power 4.
Interviewer: Okay.
Raoul: So, I guess, 2 power n.
Interviewer: Cool and can you explain, so, why does that make sense? Does it? Why do you think that's true?
Raoul: Um, it doesn't make sense to me why it has to be 2 power n. Two power n. I have no idea.

Raoul recognized the familiar numbers 8, 16, and 32 as powers of 2, a recognition based in figurative activity. He then engaged in RPG, conjecturing that an n-character password would have length 2^n, but he could not justify it combinatorially. The interviewer then asked Raoul why the number of passwords doubled from 4 to 5 characters:

Interviewer: Why would it make sense, let's say from 4 to 5 it doubles, times 2. Why would there be twice as many possibilities here as there were in the 4 case?
Raoul: That's what I'm trying to think. If I can figure that out, I would be able to find why it makes sense to be 2 power n.
Interviewer: Okay, what would you guess for a 2-character password?
Raoul: Two-character passwords? Four.
Interviewer: Okay, how about for 1-character password?
Raoul: There'd be only 2. A and B, so if it's 2, AB, AA, BA, BB. Hmm. Oh, hold up.

Reflecting on a 1-character and 2-character password, Raoul experienced an insight. He began to relate the doubling phenomenon to the combinatorial context of adding another character: "So, I noticed that this is the pattern that I got with 2 characters, so what I find that, *when I increase it to 3 characters* there will be, another character will be adding up, and that could either be A or B, so the number of passwords would be doubled." Important in Raoul's explanation is his shift of attention to what occurred when he moved from the 2-character case to the 3-character case. This marked a shift to operative activity, in that he was now coordinating the mental operation of imagining what happens when increasing the password length by one character. This also fostered PPG, as Raoul could now attend to a regularity in the process and understand why the number of passwords would double each time the length grew by 1. He was then able to explain his reasoning with the case of moving from a 3-character password to a 4-

character password: “One character has to be added up, and that character can either be A or B. So, for this one pattern, for 3 characters, there’s going to be 2 [options], there’s going to be one more pattern if I make it a 4 character.”

Raoul began by making an empirically-based generalization that the total number of passwords would be 2^n, but he could not justify his general statement. However, the development of his initial generalization was still important for Raoul’s progress. Before he wrote 2^n, the doubling aspect of the relationship was not foregrounded. Once Raoul had produced a generalization, the interviewer could then ask about doubling, which provided an opportunity for Raoul to shift to operative activity, PPG, and ultimately produce a combinatorial justification for why the number of passwords would double when adding a character.

Discussion and Implications

In this paper we have introduced a new phenomenon, empirical re-conceptualization, in which learners develop an initial conjecture based on empirical evidence or RPG, and then are able to re-conceptualize that generalization or conjecture from a structural perspective. Dr. Fisher began with an empirically-based conjecture, but that conjecture enabled her to then begin an in-depth investigation that supported an attention to how the sums changed as the summands and starting values change. She was able to reduce unfamiliar structures into familiar ones and reason in terms of general structures, both elements of structural reasoning. Raoul was similarly able to leverage his empirically-based generalization, 2^n, by shifting from RPG to PPG, and he did so by carrying out structural operations in thought, imagining what would occur when shifting from a 2-character password to a 3-character password, and again from a 3-character password to a 4-character password. Our findings indicate that empirical re-conceptualization can serve as a vehicle to transform empirical patterns into meaningful sources of verification, justification, and proof. This confirms de Villiers’ (2010) claim that “experimental investigation can also sometimes contribute to the discovery of a hidden clue or underlying structure of a problem, leading eventually to the construction or invention of a proof” (p. 215).

Certainly, students frequently identify patterns that they do not understand or cannot justify; this remains a common problem. A danger remains that students will engage in empirical investigation but then not seek to re-conceive their resulting generalizations or conjectures structurally. Our interest is in understanding why some participants in our study were able to engage in empirical re-conceptualization, while others were not. We note that both Raoul and Dr. Fisher had mechanisms by which they could shift their attention towards structural relationships. At times this ability was spontaneous (in the case of Dr. Fisher) and at other times, it required direction from the interviewer (in the case of Raoul). This suggests that directing students towards the contextual genesis of the patterns they generalize may be an effective strategy for supporting empirical re-conceptualization. In addition, it suggests that when students engage in empirical patterning activities, it is preferable to have them do so within a particular, concrete context. Ultimately, our findings indicate that the activity of generalizing empirical patterns can serve as a bridge to more generative and productive mathematical activity.

Acknowledgments

The research reported in this paper was supported by the National Science Foundation (grant no. DRL-1419973).

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BEYOND PATTERNS: MAKING SENSE OF PATTERN-BASED GENERALIZATIONS THROUGH EMPIRICAL RE-CONCEPTUALIZATION

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Identifying patterns is an important part of mathematical reasoning, but many students struggle to justify pattern-based generalizations. Some researchers argue for a de-emphasis on patterning activities, but empirical investigation has also been shown to support discovery and insight into problem structures. We introduce a phenomenon we call empirical re-conceptualization, which is the development of a generalization based on an empirical pattern that is subsequently re-interpreted from a structural perspective. We define and elaborate empirical re-conceptualization by drawing on data from secondary and undergraduate students, and identify three major affordances: Empirical re-conceptualization can serve as (a) a source of verification, (b) a means of justification, and (c) a vehicle for generating insight.

Keywords: Reasoning and Proof, Cognition, Algebra and Algebraic Thinking

Objective: Leveraging the Power of Pattern-Based Generalizations

Recognizing and developing patterns is a critical aspect of mathematical reasoning. Many students are adept at recognizing and formalizing patterns (Pytlak, 2014), but they can also struggle to understand, explain, and justify those very patterns they develop (Čadež & Kolar, 2014). One source of students' difficulties may rest with the empirical nature of those generalizations. Students can become overly reliant on examples and infer that a universal statement is true based on a few confirming cases (Knuth, Choppin, & Bieda, 2009). One potential solution is to help students understand the limitations of empirical evidence and thus recognize the need for deductive arguments (e.g., Stylianides & Stylianides, 2009). These approaches have shown some success in helping students see the limitations of examples, but they also frame empirical reasoning strategies as stumbling blocks to overcome.

In contrast, we have identified a phenomenon that we call *empirical re-conceptualization*, in which students identify a pattern, form an associated generalization, and then re-interpret their findings structurally. From this perspective, students can bootstrap their pattern-based generalizations into mathematically meaningful insights and arguments. In this paper, we describe and elaborate the construct of empirical re-conceptualization and address the following questions: (a) What characterizes students' abilities to leverage pattern-based generalizations in order to develop mathematical insights? (b) What are the conceptual affordances of empirical re-conceptualization? We offer a secondary example, discuss the affordances experienced, and consider ways in which instruction can support the practice of empirical re-conceptualization.

The Drawbacks and Opportunities of Empirical Reasoning

While an emphasis on patterning that lacks meaning can promote the learning of routine procedures without understanding (Fou-Lai Lin et al., 2004), there are also a number of affordances that can arise from empirical investigation. The act of developing empirically-based generalizations can foster the discovery of insight into a problem's structure, which could consequently support proof development (de Villiers, 2010). The degree to which pattern generalization is an effective route to proof is an open question, but there is evidence that students can and do engage in a dynamic interplay between empirical patterning and deductive argumentation (e.g., Schoenfeld, 1986).

Students lack sufficient experience with developing meaning from patterns. Curricular materials emphasize patterning activities that end with a generalization, typically an algebraic rule; developing an associated justification is seldom emphasized in standard classroom tasks. In fact, students typically receive little, if any, explicit instruction on how to strategically analyze examples in developing, exploring, and proving generalizations (Cooper et al., 2011). We propose that empirical re-conceptualization can be one way to provide opportunities to develop mathematical insight and deductive argumentation from pattern-based generalizing activities.

Theoretical Perspectives: Structural Reasoning

Harel and Soto (2017) identified five major categories of structural reasoning: (a) pattern generalization, (b) reduction of an unfamiliar structure into a familiar one, (c) recognizing and operating with structure in thought, (d) epistemological justification, and (e) reasoning in terms of general structures. The first category further distinguishes between result pattern generalization (RPG) and process pattern generalization (PPG) (Harel, 2001). RPG is a way of thinking in which one attends solely to regularities in the result. The example Harel gave is observing that 2 is an upper bound for the sequence $\sqrt{2}$,$\sqrt{2\ +\sqrt{2}}$, $\sqrt{2+\sqrt{2+\sqrt{2}}}$, … because the value checks for the first several terms. When we refer to empirical re-conceptualization and the identification of a pattern based on empirical evidence, we are referring to RPG. In contrast, PPG entails attending to regularity in the process. Harel discussed how one might engage in PPG to determine that there is an invariant relationship between any two consecutive terms of the sequence, $a_{n+1} = \sqrt{a_n + 2}$, and therefore reason that all of the terms of the sequence are bounded by 2 because $\sqrt{2} < 2$.

We define empirical re-conceptualization as the process of re-interpreting a generalization based on RPG from a structural perspective. By structural perspective, we mean engaging in any of the following activities: (a) shifting from RPG to PPG; (b) reducing an unfamiliar structure into a familiar one; (c) carrying out operations in thought without performing calculations; (d) forming and reasoning with a new conceptual entity; or (e) shifting from figurative to operative activity. In short, re-interpreting a generalization from a structural perspective entails the ability to recognize, act upon, and reason with general structures.

Methods

Barney (a 7th-grade student) and Homer (a 9th-grade student) participated in a paired teaching experiment (Steffe & Thompson, 2000), which took place across five sessions averaging 75 minutes each. An aim of the teaching experiment was to investigate the students' generalizations about the areas and volumes of growing figures, and then to study their development of combinatorial reasoning by exploring the growing volumes of hypercubes and other objects in 4 dimensions and beyond.

All teaching sessions were videoed and transcribed. We first drew on Ellis et al.'s (2017) RFE Framework to identify generalizations, and then used open coding to infer categories of generalizing activity based on the participants' talk, gestures, and task responses. We then identified an emergent set of relationships between the participants' patterning activities and the types of generalizations they formed; this yielded the category of empirical re-conceptualization.

In a final round we re-visited the data corpus in order to identify all instances of empirical re-conceptualization, the generalizations that led to each instance, and the subsequent explanation or justification. In this manner we were able track the changes in students' activity after engaging in re-conceptualizing, which led to the identification of the affordances detailed below.

Results

We found three major affordances of engagement in empirical re-conceptualization. Namely, empirical re-conceptualization can serve as (1) a source of *verification*, (2) a means of *justification*, and (3) a vehicle for *generating insight*. Within the third category, we identified three types of insight: (3a) re-interpretation within a different context or representational register, (3b) the creation of a new generalization, and (3b) the establishment of a new piece of knowledge. In order to characterize the phenomenon of empirical re-conceptualization and its associated affordances, we present an exemplar case.

Secondary Case: Growing Volumes in Three Dimensions and Beyond

Barney and Homer explored the added volumes of three-dimensional, four-dimensional, and other n-dimensional "cubes" that grew uniform amounts in every direction. They began by determining the added volume of an n by n by n cube that grew 1 cm in height, width, and length. The students worked with physical cubes to consider the component pieces and determined that the added volume would be $3n^2 + 3n + 1$. When they then investigated the added volume of a cube that grew x cm in each direction, the students simply generalized from their prior result. Homer wrote "$(3x)n^2 + (3x)n + x^2$", replacing the 3 in the first two terms of his original expression with a $3x$, and replacing the 1 in the last term, which he had conceived as 1^2, with an x^2. Unsure about the correctness of this expression, Barney said, "let me model on the cube", which he used to verify that the first term, $3xn^2$, was correct because it represented three additional rectangular prisms, each with a volume of xn^2. Both students then realized errors in the next two terms. Barney explained that the second term should actually be $3x^2n$ "because you're adding 3 of x by x by n." Both students also realized the final term would have to be x^3.

The students' original generalization was based on the *result* of their prior activity in building up additional volume components, rather than attending to the *process* by which they grew the cube's volume. However, Barney then experienced a need to verify Homer's result, which led to re-conceptualizing the generalization within the context of volume. He took the algebraic structure and made sense of it geometrically, in the process coordinating his mental activity of constructing component volumes and translating those quantities to algebraic representations.

The students eventually went on to determine expressions of added volume for the 2nd, 3rd, and 4th dimensions, which the teacher-researcher wrote in Figure 1. Homer then saw a pattern in the expressions, exclaiming, "Oh, I know what's happening!":

Homer: It is simple, as 2 – sorry I'm writing on it. [Begins to draw the blue lines.] Two plus 1 is 3, and 2 plus 1 is 3, 3 plus 3 is 6, 3 plus 1 is 4, 1 plus 3 is 4. [Writes the red numbers.]

TR: Whoa. Huh.

Barney: Wow. It's just that one triangle, Pascal's triangle, right?

Homer recognized the pattern in which each coefficient could be determined by adding the sum of the coefficients of the prior consecutive terms. Pascal's triangle then became a mechanism for determining the additional volume of a 5th-dimensional solid, which the students wrote as "$5n^4 + 10n^3 + 10n^2 + 5n^1 + 1^5$". They then decided to check their answer by listing the arrangements of three ns and two 1s (the $10n^3$) case, which served to verify that the coefficient was indeed 10. Barney then realized that given that they had verified the $10n^3$ case, they did not need to check the $10n^2$ case: “We can basically just take this and switch all the *n*s to 1s and 1s to *n*s.” This explanation of symmetry caused Homer to then extend that finding to new cases: “Oh, and you know what? You can do the same for these (pointing to the $5n^4$ and the $5n^1$ terms)…you can just replace these 1s for *n*s.”

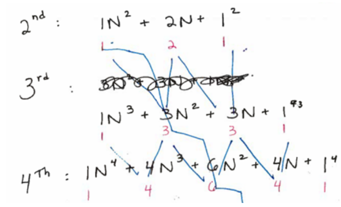

Figure 1: Expressions for added volume in the 2nd, 3rd, and 4th dimensions

Homer and Barney initially developed a generalization based on Pascal’s triangle, which allowed them to determine the expression for added volume. Their subsequent listing activity enabled the students to re-interpret that expression combinatorially. That pattern allowed the students to engage in a verification process and subsequently reason about outcomes to develop a new insight, that there must be symmetry in the coefficients. Barney was able to reflect on his operations in listing the ten outcomes and realize that there was nothing special about the characters *n* and 1, and that they could simply be reversed in the case of determining the combinations of two *n*s and three 1s. This then supported Homer’s new generalization.

Discussion

Empirical re-conceptualization can serve as a source of *verification*, such as when Barney checked the algebraic expression for adding *x* cm to a cube by appealing to the notion of volume. It can also serve as a source of *justification*, which we saw when Barney justified Homer’s pattern of *x*s in the expression $3xn^2 + 3x^2n + n^3$. We also saw the students developing *insight*. They developed new knowledge and understanding, such as when Barney generated the idea that the coefficient of n^3 must be identical to the coefficient of n^2, which then supported Homer’s ability to establish a new generalization that could be extended to the other terms, $5n^4$ and $5n$.

These affordances suggest that empirical re-conceptualization can serve as a vehicle to transform empirical patterns into meaningful sources of verification, justification, and insight. Certainly, students may also identify and generalize patterns that they do not understand or cannot justify. A danger is that students will engage in empirical investigation but then not seek to re-conceive their findings structurally. We find it useful to explore the conditions that can best support students’ transition to the productive next step, that of empirical re-conceptualization. Our data suggest that directing students back towards the contextual genesis of the patterns they generalize may be an effective strategy for supporting empirical re-conceptualization. With the support of concrete contexts for meaning making, the activity of generalizing empirical patterns can serve as a bridge to more generative and productive mathematical activity.

Acknowledgments

The research reported in this paper was supported by the National Science Foundation (grant no. DRL-1419973).

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Empirical Re-conceptualization as a Bridge to Insight

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Abstract: Identifying patterns is an important part of mathematical investigation, but many students struggle to justify their pattern-based generalizations. These findings have led some to argue for a de-emphasis on patterning, but others argue that it can support insight into a problem's structure. We introduce a phenomenon, empirical re-conceptualization, in which learners generalize based on an empirical pattern, and then re-interpret it from a structural perspective. We elaborate this construct by providing a representative example of empirical re-conceptualization from two secondary students. Our findings indicate that developing empirical results can foster subsequent insights, which can in turn lead to justification and proof.

Introduction: The affordances and constraints of empirical investigation

Developing patterns is a key aspect of mathematical activity, but students are often discouraged from relying on empirical evidence to defend mathematical claims. Researchers posit that a primary source underlying students' struggles to justify their conjectures concerns their treatment of empirical evidence. Students can be overly reliant on examples and often infer that a mathematical statement is true on the basis of checking a small number of cases (Knuth, Choppin, & Bieda, 2009). Students can be adept at leveraging patterns in order to develop generalizations, but then struggle to understand, explain, and justify their results (Čadež & Kolar, 2014). One potential solution is to help students understand the limitations of empirical evidence as a means of mathematical justification and thus recognize the need for proof (e.g., Stylianides & Stylianides, 2009). These approaches have shown some success in helping students learn the limitations of examples, but they also frame empirical reasoning strategies as stumbling blocks to overcome.

In contrast, we have identified a phenomenon that we call *empirical re-conceptualization*, in which students identify empirical patterns, form associated generalizations, and then re-interpret their findings from a structural perspective. Rather than positioning example-based reasoning as an unsophisticated approach to de-emphasize, we identify the ways in which students can bootstrap empirical reasoning into mathematically meaningful insights. In this paper, we address the following questions: (a) What characterizes students' abilities to leverage empirical patterns in order to develop mathematical insights? (b) What are the conceptual affordances of engaging in empirical patterning activity? We describe and elaborate this construct with an example with secondary students, and discuss the finding that developing results from empirical patterns can serve as a launch point for subsequent insight, including verification, justification, and proof.

The interplay between empirical reasoning and deductive reasoning

It is generally recognized that students' arguments are expected to progress from empirically-based justifications to deductive proofs. Indeed, various reasoning hierarchies have been proposed that reflect this expected progression (e.g., Balacheff, 1987). Although these hierarchies delineate levels of increasing sophistication in students' arguments, they do not sufficiently account for how students' empirical reasoning will make the transition to deductive reasoning. Indeed, many students find that transition challenging to navigate, and this is a challenge that persists even at the undergraduate level (Stylianides & Stylianides, 2009).

Despite these potential drawbacks, many researchers also point to the affordances of empirical investigation. Developing empirically-based generalizations can support the discovery of insight into a problem's underlying structure, which can, in turn, foster proof construction (de Villiers, 2010). Students can and do engage in a dynamic interplay between empirical patterning and deductive argumentation (Küchemann, 2010). de Villiers (2010) noted that mathematicians regularly engage in experimentation and deduction as complementary activities, and Tall et al. (2008) argued that students can bootstrap their empirical investigations into more sophisticated knowledge structures. It may be that students become stuck in a focus on empirical relationships because they lack sufficient experience with developing structural meaning from patterns. Curricular materials emphasize patterning activities that begin and end with the development of a generalization, typically presented as an algebraic rule. Forming a connection between the generalization and a structural justification for its reasonableness is seldom emphasized in standard classroom tasks. However, given the above evidence that meaningful connections can be developed from empirical investigation, we advocate for positioning empirical patterning as a bridge to insight and deduction.

Structural Reasoning and Figurative versus Operative Activity

Considering structure to be something made up of a number of parts that are held together in a particular manner, Harel and Soto (2017) introduced five major categories of structural reasoning: (a) pattern generalization, (b) reduction of an unfamiliar structure into a familiar one, (c) recognizing and operating with structure in thought, (d) epistemological justification, and (e) reasoning in terms of general structures. The first category further distinguishes between two types of generalizing: Result pattern generalization (RPG) and process pattern generalization (PPG) (Harel, 2001). RPG is a way of thinking in which one attends solely to regularities in the result. The example Harel gave is observing that 2 is an upper bound for the sequence $\sqrt{2}$,$\sqrt{2+\sqrt{2}}$, $\sqrt{2+\sqrt{2+\sqrt{2}}}$, … because the value checks for the first several terms. When we refer to empirical re-conceptualization and the identification of a pattern based on empirical evidence alone, we are referring to RPG. In contrast, PPG entails attending to regularity in the process. To extend the above example, Harel discussed how one might engage in PPG to determine that there is an invariant relationship between any two consecutive terms of the sequence, $a_{n+1} = \sqrt{a_n + 2}$, and therefore reason that all of the terms of the sequence are bounded by 2 because $\sqrt{2} < 2$.

We also draw on a distinction between forms of mental activity. Figurative activity involves attending to similarity in perceptual or sensorimotor characteristics (Piaget, 2001). For instance, one could associate the sine curve with circular motion through conceiving both the graphical register and the physical register as representations of smooth continuous motion (Moore et al., in press). In contrast, operative activity entails attending to similarity in structure or function through the coordination and transformation of mental operations. To continue the above example, a student could associate the sine curve with circular motion through conceiving both as representing an invariant relationship of co-varying quantities. A shift from RPG to PPG is often accompanied by a shift from figurative to operative mental activity, and we consider operative activity to be a hallmark of the ability to reason structurally. We therefore define empirical re-conceptualization as the process of re-interpreting an empirically-determined generalization from a structural perspective, which can include engaging in any of the five major categories of structural reasoning, with the exception of the RPG sub-category of pattern generalization, or shifting from a figurative to an operative association.

Methods

We conducted a paired teaching experiment over 5 sessions, with each session lasting between 30 and 90 minutes. Homer was a 9th-grade student who had completed Algebra I, and Barney was a 7th-grade student who had completed pre-algebra. Our aim was to investigate the students' conjectures and generalizations about the areas and volumes of growing figures, and then to investigate their development of combinatorial reasoning by exploring the growing volumes of hypercubes and other objects in 4 dimensions and beyond.

All teaching sessions were videoed and transcribed. Using the constant-comparative method, we analyzed the data in order to identify the participants' generalizations and to characterize the mental activity that fostered them. For the first round of analysis we drew on Ellis et al.'s (2017) RFE Framework, and in subsequent rounds we used open coding to infer categories of generalizing based on the participants' talk, gestures, and task responses. The next round of analysis supported the development of an emergent set of relationships between the participants' patterning activities and their generalizations; this yielded the emergent category of empirical re-conceptualization. In a final round we re-visited the data corpus in order to identify all instances of empirical re-conceptualization and the initial generalizations that led to each instance. In this manner we were able to determine the characteristics of empirical re-conceptualization and track the changes in students' generalizing after engaging in re-conceptualizing, which led to the identification of the affordances detailed below.

Results

In order to characterize the participants' abilities to leverage empirical patterns to develop mathematical insights, we present an exemplar case. Barney and Homer initially worked with the following task: "Say you have an n by n by n cube, and you add 1 cm to the height, width, and length. What is the added volume of the new cube?" Both students worked with blocks to think about the component pieces of a larger cube and reason that the added volume would be $3n^2 + 3n + 1$. The teacher-researcher then gave the same task for a four-dimensional n by n by n by n hypercube. In order to introduce combinatorial reasoning as a way to ground sense-making, she asked the students to consider the dimensions height, width, length, and a fourth dimension introduced as "slide". Homer and Barney determined that when adding 1 cm to the n by n by n slice of a hypercube, they would have four outcomes: a) $1 \times n \times n \times n$; b) $n \times 1 \times n \times n$; c) $n \times n \times 1 \times n$; and d) $n \times n \times n \times 1$. Both students could then justify why the n term and the n^3 term should have a coefficient of 4. The students then re-wrote their expressions to be "$4n^3 + 4n^2 + 4n + 1^4$". This expression is incorrect – the coefficient for the n^2 term is not 4 – but it is an understandable generalization. It was a result of Barney and Homer generalizing from their operative activity of determining the coefficient of n^3 to be 4, as well as from a figurative extension of the numeric structure of the three-dimensional case to the four-dimensional case.

The teacher-researcher then asked the students to list out all of the options for the n^2 coefficient. They correctly listed 6 options for arranging two n's and two 1's into four slots. By this point, the students had now correctly determined expressions for the second, third, and fourth dimensions, and the teacher-researcher wrote down their expressions in Figure 1. The expressions caused Homer to have a realization:

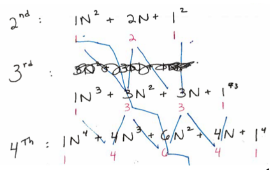

Figure 1. Expressions for added volume in the 2nd, 3rd, and 4th dimensions

Homer: I know what is happening Barney. It is simple, as 2 – sorry I'm writing on it. [Begins to draw the blue lines]. Two plus 1 is 3, and 2 plus 1 is 3, 3 plus 3 is 6, 3 plus 1 is 4, 1 plus 3 is 4. [Writes the red numbers into the figures.]

TR: Whoa. Huh.

Barney: Wow. It's just that one triangle, Pascal's triangle, right?

Homer recognized a pattern, and he knew that each coefficient for each term could be determined by adding the sum of the coefficients of the consecutive terms from the prior dimension. This was an empirically-based generalization. Pascal's triangle then became a mechanism for determining an expression for the additional volume of a fifth-dimensional figure. Homer said, "Right here, I'm going to write down what it would be if it followed the sequence." He and Barney both wrote "$5n^4 + 10n^3 + 10n^2 + 5n^1 + 1^5$", and then they decided to check their answer by listing out the arrangements of three ns and two 1s (the $10n^3$) case, which served to verify that the coefficient was indeed 10. As the students reflected on their activity, Homer explained that he did not want to do the tedious listing that would be required to double check each coefficient: "I don't want to check all of these. I was just going to check one, to kind of, maybe I'll check two or something." Barney then realized that given that they had verified the $10n^3$ case, they did not need to check the $10n^2$ case: "We can basically just take this and switch all the ns to 1s and 1s to ns." Explaining further, Barney said, "It will be the same combinations here, just substituting I for n and n for I" (he inadvertently starting calling the 1s "I"s). This explanation of symmetry caused Homer to then extend that finding to new cases: "Oh, and you know what? You can do the same for these (pointing to the $5n^4$ and the $5n^1$ terms)…you can just replace these 1s for ns."

Homer and Barney initially developed a generalization based on the empirical recognition of Pascal's triangle. This empirically-based generalization then provided them with a conjecture for the expression of the added volume in the fifth dimension, which they could then check through listing that there were indeed ten combinations of three ns and two 1s. Once they had confirmed that coefficient, they looked back at their conjectured expression and realized that they would again need to list ten outcomes to check the other coefficient. This sparked a desire to avoid repeating the listing process, which motivated a justification for symmetry. Barney was able to explain why the coefficients for the n^3 term and the n^2 term must be the same, which Homer could then extend to the n^4 and n terms. Homer was therefore able to re-interpret combinatorially what he had first conjectured using only the patterns in Pascal's triangle. The numerical pattern, which was developed from RPG, allowed the students to engage in a verification process and subsequently reason about outcomes to justify the symmetry in the coefficients. Barney's reasoning in particular was grounded in operative activity: He was able to reflect on his coordination of operations in listing the ten outcomes and realize that there was nothing special about the characters n and 1, and that they could simply be reversed in the case of determining the combinations of two ns and three 1s.

Discussion and implications

We have introduced a new phenomenon, empirical re-conceptualization, in which learners develop an initial generalization based on empirical evidence and then are able to re-conceptualize it from a structural perspective. The students both carried out structural operations in thought by justifying why it would be legitimate to replace *n*s with 1s for different listing options. Our findings indicate that empirical re-conceptualization can serve as a vehicle to transform empirical patterns into meaningful sources of verification, justification, and proof. This confirms de Villiers' (2010) claim that "experimental investigation can also sometimes contribute to the discovery of a hidden clue or underlying structure of a problem, leading eventually to the construction or invention of a proof" (p. 215).

Certainly, students may also identify and generalize patterns that they do not understand or cannot justify; this remains a common phenomenon. A danger is that students will engage in empirical investigation but then not seek to re-conceive their findings structurally. We find it useful to explore the conditions that can best support students' transition to the productive next step, that of engaging in empirical re-conceptualization. Homer and Barney had mechanisms by which they could draw their attention back to a combinatorial context. Even though they developed empirically-based generalizations, those statements were never far from their understanding of the combinatorial situations. This suggests that directing students back towards the contextual genesis of the patterns they generalize may be an effective strategy for supporting empirical re-conceptualization. Rather than discouraging reliance on empirical patterns or requiring students to prematurely shift to abstracted representations, we suggest situating instruction within particular, concrete contexts that can provide a meaningful foundation for empirical re-conceptualization. For Homer and Barney, this context was combinatorial; in other cases, we have found that contexts that leverage students' engagement with real-world quantities, such as distance, time, speed, length, area, and volume, can similarly provide fruitful supports for understanding and justifying conjectures. With the support of concrete contexts for meaning making combined with instructional moves that encourage students to consider their empirical findings in light of those contexts, our findings indicate that the activity of generalizing empirical patterns can serve as a bridge to more generative and productive mathematical activity.

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Acknowledgments

The research reported in this paper was supported by the National Science Foundation (grant no. DRL-1419973).

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PK 12

Why Multiply? Area Measurement and Multiplicative Reasoning

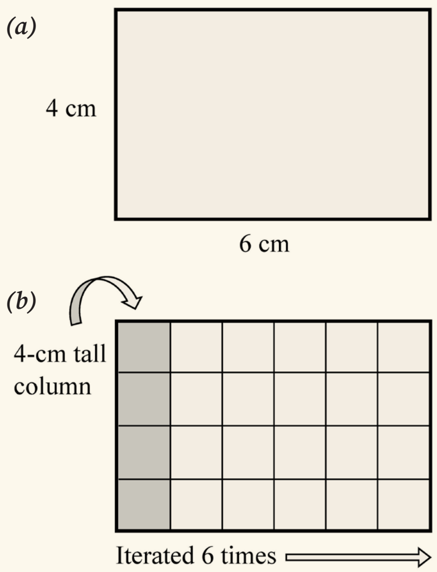
Asked to quantify the changes in area of growing rectangles, these students reasoned about multiplicative relationships in interesting new ways.

Brandon K. Singleton and Amy B. Ellis

Think about a rectangle that is 6 cm long and 4 cm tall (see figure 1a). Why can the rectangle's area be found by multiplying its length by its width? Despite our best efforts, helping children understand *why we multiply* to measure area is an enduring problem. Many students can recite the $A = L \times W$ formula, but they often do not understand it. Making sense of arrays of unit squares is one basis for understanding the transformation of length measurements into area measurements. For instance, for a 4-cm by 6-cm rectangle, students can think about the area as measured in square units, and the total number of square units, 24, can be found by imagining a column of 4 square centimeters iterated 6 times (see figure 1b).

Some students do develop this meaning. For instance, Anya, a sixth grader, calculated the area of a 1.5-cm by 4-cm rectangle and explained that multiplying made sense because there would be 1.5 rows and 4 columns: "That creates sort of, like, squares or

Fig. 1



For a 4-cm by 6-cm rectangle partitioned into columns and rows, students can think about measuring the area in square units (a); and finding the total by imagining a column of 4 square centimeters iterated 6 times (b).

rectangles within the rectangle, and there's 4 rows of 1.5. So that would be like 4 times 1.5, so that's how we figure out how many, or how the area would be."

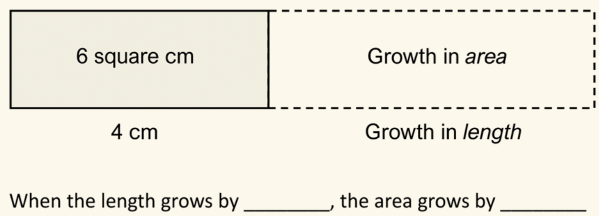
Not all students, however, are able to make sense of the row-column array structure in the way Anya did. For this reason, Battista and colleagues suggest that typical instructional treatments of area and multiplication should be rethought: "If students do not see a row-by-column structure in these arrays, how can using multiplication to enumerate the objects in these arrays, much less using area formulas, make sense to them?" (Battista et al. 1998, p. 531). Students may not naturally group the unit squares into rows and columns, and if they do, they may not associate the number of rows and columns with corresponding side measures. Moreover, imagining partial rows and columns is difficult for rectangles with noninteger dimensions. Many students we observed did not make the argument Anya provided for the 1.5-cm by 4-cm rectangle.

We designed an activity called the Growing Rectangle problem that connects multiplication with area measurement in a new way. Inspired in part by a task given by Johnson (2013), the Growing Rectangle problem has three key features:

1. Areas are calculated by imagining a *transformation* rather than by measuring a fixed object.
2. The setup provides a *length-area pair* instead of a length-height pair.
3. The prompt allows for *multiple initial* responses as students explore how changes in length and area occur together (see figure 2).

We used the Growing Rectangle problem with 13 middle-school students in grades 6–8. Although nine students eventually used the area formula, most did not think of the formula immediately. Instead, the students had to grapple with how length and area change together. We share the thinking of several students below.

Fig. 2



The Growing Rectangle problem

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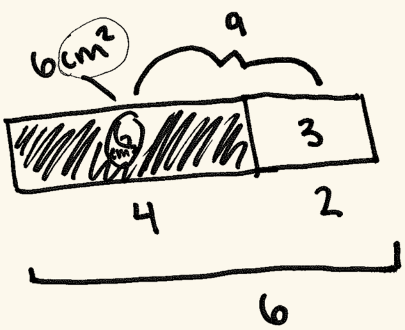
doi: 10.5951/MTLT.2019.0063

OLIVIA: UNIT RATE

Olivia, a seventh grader, filled in the blanks with 2 and 3 and explained, "I know that 2 is half of 4, so then I did half of 6." Olivia's picture in figure 3 shows a growth of 2 cm in length and 3 cm^2 in area. She also developed other pairs, such as 3 cm: 4 1/2 cm^2, 1 cm: 1 1/2 cm^2, and 16 cm: 24 cm^2. In explaining her work for the 3 cm: 4 1/2 cm^2 pair, Olivia said, "I reduced it to be, like, the unit rate of how much it would go up by. So, if I know that 1 [centimeter in length] would be 1 1/2 [centimeters squared in area], then I times that by 3, to get 4 1/2." Olivia recognized that the rectangle gained area at the rate of 1 1/2 cm^2 for every additional 1 cm of length, and she used that rate to find other pairs.

Olivia could also express this relationship as a general formula in terms of expressing the growth in area for an *x*-cm growth in length. She wrote "1.5*x*" and explained, "You can replace *x* with anything." Olivia did not think about 1.5 as the rectangle's height at this point; instead, it was a unit rate expressing how much area the rectangle gains per 1 cm of length as it grows (see figure 4).

Fig. 3



Olivia's picture of the rectangle growing by 2 cm in length and 3 cm^2 in area.

CONNECTING MULTIPLICATION TO AREA UNITS

In geometric measurement, it is important to understand what the unit of measure is. The Growing Rectangle problem gives measurements, but it does not indicate what the units that were used for measuring look like. Students must bring their own meanings to the problem to infer what a square centimeter is and where the 6 cm^2 are in the figure. Teachers can help

Fig. 4

$$\underset{\substack{\text{Area} \\ \text{cm}^2}}{A} = \underset{\substack{\text{Rate} \\ \text{cm}^2\text{ per 1} \\ \text{cm length}}}{1.5} \cdot \underset{\substack{\text{Length} \\ \text{cm}}}{x}$$

The meaning for Olivia's area formula

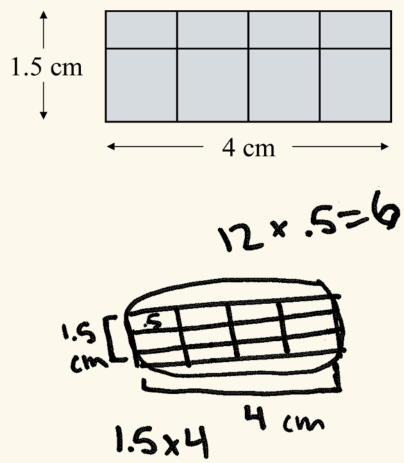
students make their ideas explicit and think more purposefully about units of measure. We asked students to show where the 6 cm^2 were in the rectangle. This can be challenging when using unit squares because the height is not a whole number. It requires conceiving of the 1.5-cm height as 1 square centimeter and 1/2 of a square centimeter, iterated 4 times to create 6 square centimeters (see figure 5a).

In Olivia's case, she partitioned the rectangle into 4 columns and 3 rows to make 12 rectangular cells. She understood that each cell was 0.5 cm tall to make an area of 0.5 cm × 1 cm = 0.5 cm2 (see figure 5b), explaining, "The height of each of these little boxes would be 0.5. And so then, you'd just go 0.5, 1, 1.5 (the entire column), and then times 4." However, Olivia was unable to relate the cells in her drawing to square centimeters. When

Fig. 5

(a)

(b)



The 1.5 cm by 4 cm rectangle is partitioned into (a) square units; and (b) rectangular cells.

the interviewer, said, "I see 12 rectangles," and asked her, "Can you show me where the 6 would be then—the 6 square centimeters?" Olivia responded, "Well, this is kind of the 6. So, the whole thing" [she circled the entire rectangle].

Although Olivia appeared unfamiliar with the convention that 1 cm^2 is a 1-cm by 1-cm square, she showed flexibility in partitioning the rectangle and in identifying the correct size or amount that corresponded to her partition. In many tasks, students are provided with square units, such as tiles, and are asked to measure areas. They do not have to think about where the unit tiles come from. In contrast, the Growing Rectangle problem provides students with an area measure and challenges them to think about what 1 cm^2 represents. By using partitioning strategies, students can come to appreciate that a multiplicative relationship exists between the amount contained in one unit and the amount contained in the whole measure. Olivia was able to relate the amount in one of her cells to the whole rectangle by using multiplication, writing the 6-cm^2 area both as 1.5×4 and as $12 \times .5$ (see figure 5b).

SPIKE: PROPORTIONALITY VERSUS SIMILARITY

Like Olivia, Spike (a seventh grader) believed that the length and area were proportional. He found new pairs of length and area by setting up proportions and solving with cross products. However, Spike became uncertain about his answers because he realized that the growing rectangle did not maintain similarity as it grew. Spike confused similarity with proportionality, stating, "The length is expanding, but the width isn't; so that wouldn't make it proportional. That would make it a different shape." Because the rectangle did not maintain similarity, Spike came to believe that all of his answers were wrong. When he checked his answers using the area formula, he was astonished that they matched: "Oh that was right. Huh, that's weird . . . . I'm getting the same answers as I had before."

Spike is not alone in confusing proportionality and similarity. In another study, students solved a problem about a window painting of Father Christmas (see figure 6) that was to be scaled up by a factor of three (Van Dooren et al. 2003). When asked how much paint would be needed, almost all the students multiplied the original amount by three instead of nine. The students could not explain why they believed a proportional strategy worked, stating, "I don't know. I just solved it that way," and "It just works. I don't know why" (pp. 206–7).

Although Spike assumed that a proportional strategy must imply similarity, the students in the study that Van Dooren and colleagues conducted (2003) assumed that similarity must imply a proportional strategy.

Both the Father Christmas problem and the Growing Rectangle problem can cause confusion in students who have come to think of similarity and proportionality as two sides of the same coin. Typical area tasks may contribute to this confusion if all the proportion problems from geometry rely on similar figures. The Growing Rectangle problem, in contrast, provides an opportunity to work with proportionality in a geometry setting that does not rely on similarity. Teachers could use the Growing Rectangle problem as a bridge to making sense of similarity by breaking the scaling transformation into two steps. Each step involves scaling the area in one dimension by a proportional factor k, with the final result being an increase by a factor of k^2. (View video 1 for further illustration of scaling in two dimensions.)

Fig. 6

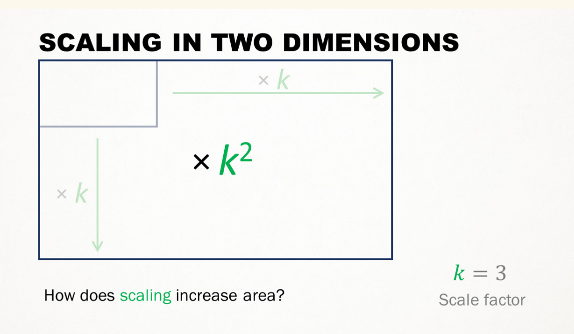
Bart is a publicity painter. In the last few days, he had to paint Christmas decorations on several store windows. Yesterday, he made a drawing of a 56 cm high Father Christmas on the door of a bakery. He needed 6 ml of paint. Now he is asked to make an enlarged version of the same drawing on a supermarket window. This copy should be 168 cm high. Approximately how much paint will Bart need to do this?

The Father Christmas problem. From "Improper Applications of Proportional Reasoning," by W. Van Dooren, D. De Bock, L. Vershaffel and D. Janssens, 2003, Mathematics Teaching in the Middle School *9, no. 4, p. 205. Reprinted with permission.*

CHALLENGES IN SUPPORTING MULTIPLICATIVE REASONING

Not all students to whom we gave the Growing Rectangle task related length and area multiplicatively. Four students approached the task using additive rather than multiplicative reasoning. For example, Willow, a sixth grader, wrote, "If the length and the area grow evenly (by the same amount), the area will always be 2 more than the length." Willow declared the area to always be 2 cm greater than the length, regardless of what the length would be. Thus, she identified a constant *difference* between length and area rather than a constant *ratio*. If Willow had visualized a very long rectangle, such as a 20-cm rectangle or a 100-cm rectangle, she might have realized that the corresponding area would need to be larger than 22 cm^2 (or 102 cm^2). However, it is also possible that Willow was not imagining the rectangle's area in terms of units of measure, so this type of visualization may not have prompted her to realize that the relationship between length and area could not be additive. Another approach could be to ask Willow to double the original 4-cm long rectangle. If she drew a second rectangle next to the first one (as outlined in the dashed version in figure 2), Willow could be asked to describe the area of the second (identical) rectangle with an additional length of 4 cm. This rectangle, being a copy of the first, would have 6 cm^2 for area, which might help Willow see that the total rectangle that was 8 cm long would need to have an area of 12 cm^2.

Video 1 Scaling Two Dimensions



Watch the full video online.

Another student, Angelo (an eighth grader), also initially relied on additive reasoning, stating that a rectangle with a length of 6 cm would have an area of 8 cm^2. Angelo's answer, although incorrect, was based on trying to visually estimate the size of the dashed region in figure 2. In this case, posing doubling and halving scenarios helped Angelo shift to a multiplicative strategy. For example, given a growth in length of 4 cm, Angelo realized, "I think that it would grow by 12, because is 4 is doubling, and if that's 4 centimeters [the original length] and that's 4 centimeters [the growth in length], then that [total length] would equal 8 centimeters. And then another 6 centimeters [squared, in area] would equal 12 centimeters [squared, for the total area]." Then, when asked to imagine the rectangle growing by half its amount (another 2 cm), Angelo explained, "I think it will grow by 3 [cm^2], because 4 [cm] divided by 2 is 2 [cm], and then 6 [cm^2] divided by 2 is 3 [cm^2]." Students such as Angelo and Willow can benefit from strategies recommended for fostering initial ratio reasoning, which include attending to both quantities (length and area) together; making drawings of different-sized rectangles together with tables to keep track of length and area pairs; beginning with simpler cases such as doubling, quadrupling, and halving; and gradually moving to more difficult multipliers (Lobato and Ellis 2010).

The Ratio-And-Proportion Meaning

Because students often rely on the area formula without understanding why they should multiply, we designed the Growing Rectangle problem to encourage ratio reasoning. The Growing Rectangle problem affords a new way of explaining why we multiply when calculating area, and it helps students connect proportional reasoning to area measurement. Most problems about area in the curriculum involve static shapes, and students calculate areas by counting how many unit tiles cover the region until they understand row-column arrays and derive the formula $A = L \times W$. Students justify the formula's multiplication operation with an equal-groups meaning or an array meaning. The Growing Rectangle problem leverages a different meaning of multiplication, the ratio-and-proportion meaning. This meaning does not lend itself as readily to the $A = L \times W$ formula, but it can lead to an algebraic equation based on rate thinking. Students use multiplication directly and spontaneously to calculate changes in area. They iterate and partition ratio pairs just as they would for other equal-ratio problem contexts. Multiplication can supply a ratio meaning to the units of measure so that a

6-cm^2 object is six times as large as the unit, 1 cm^2.

As students reason about the Growing Rectangle problem, we should use caution when interpreting their proportional strategies. As we saw with Spike, students may use proportions computationally before they can justify doing so conceptually. Willow and Angelo tried out multiplicative strategies only after attempting additive ones, and it was not obvious to them at first which strategy was correct. Determining why multiplicative reasoning is appropriate on the basis of the problem's geometric relationships is important for students. Asking them to justify their strategies can help.

The Growing Rectangle problem is a challenging, nonroutine task that can promote problem solving. Strategic implementation can help individualize the task to each student's strengths and needs. See the sidebar for some implementation tips. With the appropriate supports, students can develop multiplicative strategies that they can understand, justify, and eventually connect to the area formula.

Implementation tips for the Growing Rectangle problem

• **Posing:** You can leverage the value of the task by posing sequences of questions that respond to students' thinking and lead students forward. Use students' initial responses to guide next steps and manage the problem's difficulty. You can pose different values in one blank and request the missing value. Choose numbers purposefully and think about the relationships involved in moving from one number pair to the next. Skillful sequences will help your students confront and overcome challenges. You can also extend the task with new shapes, like a growing parallelogram.

• **Ratio:** This task can help students build their ratio reasoning. Some students may need to start with easier numbers that rely on doubling or halving to create ratios. Once they can do that, you can pose values that help students build up more complicated ratios. Over time, students should develop a unit ratio and express the relationship between length and area algebraically.

• **Visuals:** In addition to providing students with pictures, consider showing the situation dynamically with technology and allowing students to drag the rectangle to see its growth. Also, ask students to draw their own figures and show the quantities they calculate in the figures they have drawn.

• **Quantities:** Make sure that students do not lose track of the quantities in the situation, such as length, height, and area. Ask them to refer to the specific units they have calculated so that they can keep track of the relationships between the associated quantities. You may need to help students distinguish carefully between added growth (such as added length and added area) and total growth.

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