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The role and use of examples in learning to prove☆

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ABSTRACT

Proof is central to mathematical practice, yet a perennial concern is that students of all ages struggle in learning to prove. Mathematics education scholars have suggested that overreliance on examples to justify the truth of statements is a contributing factor for students' difficulties. While example-based reasoning has typically been viewed as a stumbling block to learning to prove, we view example-based reasoning as an important object of study and posit that examples play both a foundational and an essential role in the development, exploration, and understanding of conjectures, as well as in subsequent attempts to develop proofs of those conjectures. In this paper, we provide an overview of our project whose goals were to (a) investigate the nature of middle school and high school students', undergraduate students', and mathematicians' thinking about the examples they use when developing, exploring, and proving conjectures; and (b) investigate ways in which thinking about and analyzing examples may facilitate the development of students' learning to prove.

1. Proof in school mathematics

Proving is central to the practice of mathematics and plays an important role in learning mathematics; not surprisingly, both mathematics education scholars (e.g., Knuth, 2002; Stylianides et al., 2016; Stylianides, 2017; Yackel & Hanna, 2003; Zaslavsky, Nickerson, Stylianides, Kidron, & Winicki, 2012) and reform initiatives (e.g., Council of Chief State School Officers [CCSSO], 2010; National Council of Teachers of Mathematics [NCTM], 2000; RAND Mathematics Study Panel, 2002) have increasingly called for proof to play a more central role in the mathematics education of students at all grade levels. Indeed, both the *Common Core State Standards for Mathematics* and the *Principles and Standards for School Mathematics* espouse similar messages with regard to proving-related activities in school mathematics:

> One hallmark of mathematical understanding is the ability to justify. … Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. (CCSSO, pp. 4–6).
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> The mathematics education of pre-kindergarten through grade 12 students should enable all students to recognize reasoning and proof as fundamental aspects of mathematics, make and investigate mathematical conjectures, develop and evaluate

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mathematical arguments and proofs, and select and use various types of reasoning and methods of proof. (NCTM, p. 56).

Yet, despite the increased attention and emphasis being placed on proof in school mathematics, students of all ages continue to struggle learning to prove (e.g., Harel & Sowder, 1998; Healy & Hoyles, 2000; Knuth, Choppin, & Bieda, 2009; Stylianides et al., 2017).

2. The transition from inductive to deductive reasoning: a paradigm shift[1]

It is generally accepted that students' understandings of mathematical justification are "likely to proceed from inductive toward deductive and toward greater generality" (Simon & Blume, 1996); that is, students' justifications are expected to progress from empirical (example-based) arguments to proofs. Indeed, various mathematical reasoning hierarchies have been proposed that reflect this anticipated progression (e.g., Balacheff, 1987; Bell, 1976; van Dormolen, 1977; Waring, 2000). This anticipated progression is also often reflected in school mathematics curricular programs; for example, a widely used middle school mathematics curricular program in the United States, the *Connected Mathematics Program*, states that "Informal reasoning evolves into more deductive arguments as students proceed from grade 6 through grade 8" (Lappan, Fey, Fitzgerald, Friel, & Phillips, 2002). Although the mathematical reasoning hierarchies and curricular expectations both emphasize the anticipated inductive-to-deductive progression in students' reasoning, they fail to sufficiently account for how students actually navigate the transition. And, in fact, many students struggle to successfully navigate the transitional "leap" from inductive-to-deductive reasoning.

Researchers have suggested that students' treatment of examples and, in particular, their overreliance on examples as a means of justification, is a primary source of their difficulties in navigating the transition. Accordingly, researchers have advocated that instructional approaches be designed to help students learn the limitations of examples, and thus recognize the need for proof (e.g., Sowder & Harel, 1998; Stylianides & Stylianides, 2009; Zaslavsky et al., 2012). Such instructional approaches may indeed help students learn the limitations of examples as well as the need and an appreciation for proof, yet students nonetheless continue to struggle in learning to prove, and teachers as well struggle to facilitate the development of their students' learning to prove (e.g., Bieda, 2011; Bieda et al., 2014; Cirillo, 2011; Stylianides et al., 2013).

Although we acknowledge the limitations of examples as a means of justification as well as the need for students to move beyond example-based justifications, we do not view example-based reasoning as a stumbling block to quickly overcome. In contrast, we view example-based reasoning as playing both a foundational and an essential role in the development, exploration, and understanding of conjectures, as well as in subsequent attempts to develop proofs of those conjectures. Moreover, we posit that instructional approaches designed to help students learn to strategically think about and productively use examples in proving-related activities will facilitate the development of their learning to prove. In fact, we believe that students' failure to strategically think about and analyze examples when engaging in proving-related activities is a critical contributing factor to their difficulties in learning to prove.

3. The role of examples in proving-related activities

Examples often play a critical role in mathematicians' proving-related activities as the time spent thinking about and analyzing examples can provide not only a deeper understanding of a conjecture, but also insight into the development of a proof. As Epstein and Levy (1995) contend, "Most mathematicians spend a lot of time thinking about and analyzing particular examples," and they go on to note that "It is probably the case that most significant advances in mathematics have arisen from experimentation with examples" (p. 6). To that end, research has noted a number of traits or approaches that characterize mathematicians' example use during proving-related activities, including a metacognitive awareness of the relationship between their example use activities and proving-related activities (Lockwood et al., 2016); a complex, non-linear approach of engagement in example use during proving-related activities (Alcock & Inglis, 2008; Antonini, 2006; Weber, 2008); and a systematic, deliberate, and reflective approach to example use during proving-related activities (Lockwood, Ellis, & Knuth, 2013; Weber & Mejia-Ramos, 2011).

The role examples play in the work of mathematicians stands in stark contrast, however, to the role examples typically play in the work of middle and high school (as well as undergraduate) mathematics students. The contrast between the example use of mathematicians and of students is not surprising given that students at all grade levels typically receive very little, if any, explicit instruction on how to strategically think about and analyze examples during proving-related activities. And, in fact, students often fail to strategically think about and analyze examples (Cooper et al., 2011; Knuth, Kalish, Ellis, Williams, & Felton, 2011) and their use of examples during proving-related activities does not always turn out to be productive (e.g., Iannone, Inglis, Mejia-Ramos, Simpson, & Weber, 2011; Sandefur et al., 2013). As Iannone and colleagues noted, "When sophisticated mathematicians generate examples, they may do so in such a way that they do indeed make gains in learning about a novel concept. Our studies may indicate that the forms of example generation which less sophisticated learners undertake do not lead to such gains" (p. 10). They also contend "there is clearly a need for further empirical research in this area if we are to determine whether and how example generation tasks can lead to significant learning gains" (p. 11).

The preceding discussion highlights the critical need to better understand both the nature and evolution of example use across grade levels (middle school to undergraduate) and levels of expertise (novices to experts), and the nature of instructional practices

[1] It may be worth clarifying the meaning of several terms used throughout the paper: *inductive*, *empirical*, *deductive*, and *proof*. We use inductive and empirical interchangeably to refer to example-based arguments. We also use deductive and proof interchangeably to refer to arguments comprised of a series of logically connected assertions that one makes to justify a mathematical claim (note that the rigor of such arguments may vary across different contexts and communities).

designed to help students become more deliberate and strategic in their use of examples during proving-related activities. Yet, very little research has focused on students' thinking about and use of examples during proving-related activities. Alcock and Inglis (2008) argue that such studies are needed in order to effectively develop instructional practices that foster the development of students' learning to prove.

4. Investigating the role of examples in learning to prove: the examples project

The Examples Project, the focus of this special issue, sought to address the aforementioned void in the research and, in particular, to better understand the role examples play in the development, exploration, and justification of mathematical conjectures, with the overarching goal being to foster students' abilities to develop mathematical proofs. The project had two primary objectives:

(1) Investigate the extent to which middle school and high school mathematics students, undergraduate mathematics students, and mathematicians use examples, and study the nature of their thinking about the examples they use when developing, exploring, and proving mathematical conjectures.
(2) Investigate the ways in which thinking about and analyzing examples may facilitate the development of middle school, high school, and undergraduate mathematics students learning to prove.

In the narrative that follows, we briefly describe details regarding the 4-year project, including information about the study participants, the data that were collected, and the overarching analytic framework developed for making sense of example use during proving-related activities. Details about data analyses, the development and application of the example use framework, and major project findings are discussed in the papers that follow (the papers are briefly introduced at the end of this paper).

4.1. Study participants

There were four groups of participants in this study, representing various grade levels and levels of mathematical expertise. The four groups consisted of twelve middle school students, sixteen high school students, ten undergraduate mathematics students, and thirteen research mathematicians.

4.2. Data collection

The primary data were collected from hour-long semi-structured, task-based clinical interviews conducted with participants in each group, with the exception of the mathematicians who participated in two such interviews (13 participated in the first interview, and 10 of 13 participated in a second interview). All interviews were recorded and transcribed, and any written work was also collected. The purpose of the interviews was to characterize the ways in which mathematics students and mathematicians reason with and make use of examples as they develop conjectures, make sense of conjectures, and attempt to justify the truth of conjectures. In the case of the mathematicians, the purpose of the interviews was essentially the same, however, the interview questions were spread across two interviews.

The interviews were designed to provide insight into both the participants' uses of examples to make sense of and attempts to prove interviewer-provided conjectures (see Table 1, sample item 1) and their uses of examples in developing their own conjectures and subsequent attempts to prove the conjectures (see Table 1, sample item 2; although this item has a similar structure as sample item 1, in the process of engaging with the item new conjectures were expected to arise). The conjectures presented in the interviews often varied in terms of their "truth type"—*always true*, *sometimes true*, and *never true* in order to allow for a rich range and manifestation of example use. The items designed to elicit conjecture development typically focused on novel ideas (cf. Buchbinder & Zaslavsky, 2009; Iannone et al., 2011).

Table 1
Sample interview items.

Sample Item 1 Trevor came up with a conjecture that states: If you multiply any three consecutive numbers together, the answer will be a multiple of 6. What do you think of Trevor's conjecture? Why?
Sample Item 2 Tyson came up with a conjecture about consecutive numbers that states: For any $k > 2$, if you add k consecutive numbers together, then the sum will be divisible by k. What do you think of Tyson's conjecture? Why?
Sample Item 3 Definition. An abundant number is an integer n whose divisors add up to more than $2n$. Conjecture: A number is abundant if and only if it is a multiple of 6.

For the students, the interview items were designed to address relevant content and be accessible (i.e., understanding the underlying mathematics was not thought to be a challenge). For the mathematicians, the interview items were also designed to be accessible, in this case, to both the mathematicians (regardless of their area of expertise) and the interviewer (to allow for meaningful follow-up questions). In addition, the mathematician items were designed to be novel and challenging (see Table 1, sample item 3). Although the items presented to the mathematicians did not represent an authentic context (i.e., the nature of the conjectures presented was not of the nature of conjectures they would seek to develop or justify in the course of their practice as mathematicians), they did provide a window into the nature of their thinking as they engaged in proving-related activities.

5. Characterizing the terrain of example use during proving-related activities

One goal of the data analyses was to characterize the complexity of example use during proving-related activities through the development of a comprehensive framework that captured that complexity across a range of proving-related activities and across a wide range of mathematical expertise. To that end, we have extended prior research and have developed a comprehensive analytic framework for characterizing and making sense of the roles and uses of examples in the proving-related activities of secondary school students, undergraduate students, and mathematicians. The framework's major categories include the intended purpose an example serves, the criteria for choosing an example, the affordances that result from using the example(s), strategies of example use (both for selecting sets of examples and for using the selected examples), and the transitions or shifts in proving-related activity from example use; each major category is also delineated into sub-categories that serve to further differentiate activities within the major categories (see Table 2).

Table 2
Example Use Framework.

CATEGORY	SUB-CATEGORIES		
Purpose of Example (to…)	Understand what (the conjecture is saying)	Understand why (the conjecture is true or false)	Explore the truth domain of the conjecture
	Test the truth of the conjecture	Refute the conjecture	Confirm a belief
	Convey the claim that the conjecture is true/false	Develop a new conjecture	Convey a general argument
	Illustrate a representation	Make sense of a representation	Placate the interviewer
Criterion for choosing a particular example	Easy (to work with)	First thought of	Different from other(s)
	Minimal case	Familiar or known example	Unusual
	"Random"	Boundary case	Typical
Affordances from example use activity	Gain insight	Produce a justification	Lead to new/revised conjecture
	Generalize	Learn the limitations of examples/ appreciate need for proof	Discover an error
Strategies employed during example use activity	Selecting a set based on the diversity of examples	Searching for counterexamples	Random feature/pattern search
	Selecting a set based on the systematic variation of examples	Jumping to formality	Trying to see the examples through a structural lens
	Selecting a set based on specific mathematical properties	Building formality from examples	Informal induction
Transition/Shift	From examples to general or formal	From examples to examples	From exploration to cessation of exploration
	From general or formal to examples	From one belief about the conjecture to another	

Although going into detail about the framework is beyond the scope of this paper (it's development and application are presented in detail in the Ellis et al. paper), our rationale for presenting it here is both to highlight the complexity and diversity of example use during proving-related activities and to set the stage for the papers that follow. Throughout our data corpus, we have seen instances of example use during proving-related activities in which students spontaneously used examples (without any prompting from the interviewer), in which they used examples only after being prompted to do so by the interviewer (e.g., "Why don't you try an example"), and in which they used examples with the assistance of the interviewer (i.e., the interviewer provides an example). For the mathematicians, not surprisingly, their example use was primarily spontaneous during their proving-related activities. The framework also captures example use that may or may not have been productive in making progress toward the development of a proof; for the mathematicians, their example use was usually productive, whereas for the students, whether the example use was spontaneous, evoked, or guided, their use did not always turn out to be productive.

We view productive example use as an activity that leads to a deeper understanding of a conjecture and the underlying mathematics; leads to insight with regard to the development of a proof; leads to a an awareness of a generalization or structure by using a *generic example* (i.e., a specific example through which one can see the general case); leads to the generation of a counterexample; leads to the development of a new or revised conjecture; or leads to an appreciation for the need for proof. As an example, the following excerpt (see Fig. 1) from our data illustrates a use of an example that for the student ultimately serves as a generic example, allowing the student to see why the conjecture must always be true.

Task: Eric noticed that when he adds any whole number to the number that comes two before it and the number that comes two after it, the answer is always equal to three times the number he started with. Do you agree with Eric that this conjecture is always true? Why?	
Student work:	**Commentary:**
Okay, so, like 7, 7	In this case, the high school student initially tested the truth of the conjecture using a particular example.
and 2 before 7 is 5, 7 + 5 and then 2 after it is 9. 7 + 5 + 9 So that equals 21, right? 7 + 5 + 9 = 21	After writing the expression that represents the conjecture, he makes the initial insight that subtracting 2 and adding 2 to the starting number will cancel each other out. Although he does not use this insight to further his progress toward a proof, the insight could lead to such progress (e.g., $(x - 2) + x + (x + 2) = 3x$).
And so it's 7 times 3 is 21, and the point is that if you subtract 2 and you add 2, they cancel each other out. Oh, so you can move 2 from here to here, and then, it would just be 7+7+7.	He then notices that he can take 2 from the last number, 9, and add it to the second number, 5, with the result being three 7s. Once he makes the latter observation, he immediately concludes that the conjecture will always be true.
+2 7 + 5 + 9 = 21 7 × 3 = 21 So it will always work.	In this case, the student came to view the example as a particular instance of the general case—a generic example.

Fig. 1. A case of productive example use.

In contrast, the following excerpts (see Fig. 2a and b) from our data illustrate less productive use of examples. In the first case, the student's example use led him to be more convinced that the given conjecture was true, but did not help him make progress toward the development of a proof. In the second case, the student's example use led to a dead end and the cessation of his exploration of the task. Interestingly, in this latter case, some of the undergraduate students who took a similar approach, upon reaching the same impasse, decided to further explore the conjecture with examples, and were ultimately able to produce a proof as a direct result of their example use.

The preceding three excerpts illustrate aspects of the example use framework and also represent cases of example use that enabled students to move productively (or not) toward the development of a proof. Although our primary interest was in example use that turned out to be productive and the underlying thinking of such example use, there are also important lessons to be learned from the unproductive use of examples. In the former case, instances of productive example use indicate promise that such use is learnable, and in the latter case, instances of less or un-productive example use suggest the need to raise students' (and teachers') awareness of such uses. Ultimately the goal is help students learn to be more deliberate and strategic in their use of examples during proving-related activities. In fact, we posit that students' lack of instructional guidance and experience with using examples productively during proving-related activities is the overarching reason that many students fail to learn from examples. Yet, to date, very little research has focused on students' abilities to strategically think about and productively use examples as they engage in proving-related activities. And ultimately, without such research and the correspondingly informed instructional practices, most students are unlikely to develop the necessary abilities to productively use examples in learning to prove.

6. Overview of the following papers in this issue

In the papers that follow, we present and discuss results from our project that focus, in particular, on the role and use of examples in learning to prove. In Ellis et al., the example use framework (see Table 2) is elaborated in greater detail, with broad level comparisons across the participant groups being presented along with illustrative excerpts from the data. The authors also discuss the utility of the framework both for making sense of example use during proving-related activities and for thinking about supporting students in learning to prove. In Lynch and Lockwood, the authors compare the example use in the proving-related activities of mathematicians and students, and discuss interesting similarities and differences in the nature of their respective example use. Aricha-Metzer and Zaslavsky offer a characterization of the nature of productive and non-productive uses of examples for proving,

a

Task: Tyson came up with a conjecture that states: If you add any number of consecutive whole numbers together, the sum will be divisible by however many numbers you added up. Do you think the conjecture will be true for any 5 consecutive numbers? Why?	
Student work: [Student tried the following examples and checked that the sum of each was divisible by 5.] 10 + 11 + 12 + 13 + 14 (-1) + (-2) + (-3) + (-4) + (-5) 347 + 348 + 349 + 350 + 351 300 + 301 + 302 + 303 + 304 102,573 + 102,574 + 102,575 + 102,576 + 102,577 *Interviewer*: Do you think the conjecture will be true *for any five* consecutive numbers? *Student*: Probably yeah. Yeah. *Interviewer*: And you believe it is true because? *Student*: Because I did a bunch of trials that go really far into the depths of numbers, including negatives, which kind of sealed the deal for me because negatives are really different from positives.	**Commentary:** In this case, the high school student's focus seemed to be centered on confirming that the conjecture for five consecutive integers was true. His use of examples was enough to convince him that the conjecture was true because he tested the conjecture with a diverse set of examples (e.g., large, negative). And as a result, there was no need to test further examples or to justify beyond examples. The student did not seem to think about or analyze the examples in a way that might have allowed him to see, for example, structural characteristics of the examples that may have led to an insight regarding the development of a proof (e.g., that the first and last numbers are ±2 from the middle number, and the second and penultimate number are ±1 from the middle number). In this case, the example use was not productive, and did not help the student progress toward the development of a proof.

b

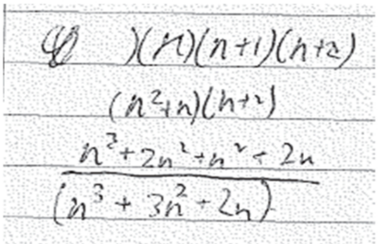
Task: Trevor came up with a conjecture that states: If you multiply any three consecutive numbers together, the answer will be a multiple of 6. Do you think Trevor's conjecture is true? Why?	
Student work: 	**Commentary:** In this case, the undergraduate student immediately represented the conjecture algebraically, expanded his initial algebraic expression, and then reached an impasse in which he was unable to make sense of the representation in relation to the claim. At this point, the student was unable to make any further progress toward a proof.

Fig. 2. a) A case of less productive example use. b) A case of example use leading to a dead end.

and draw mainly on cases in which example use did (or did not) lead to critical shifts in progress toward the development of proofs. Finally, Ozgur et al. also examine the nature of productive and non-productive uses of examples for proving, but focus on patterns of example use activity among those students who were successful in developing proofs and among those students who were not successful. In the closing paper of this special issue, John Mason, as a project outsider and expert in the area of example use in mathematics, provides a commentary about the collection of project papers, including insights gained from the work, implications for instruction, and suggestions for future research.

As Stylianides, Styliandies, and Weber (in press) state in their review of current research in the area of proof, research that offers theoretical and practical knowledge is sorely needed: "What is urgently needed are ways to help students realize the benefits of employing diagrams and examples. Research comparing the ways that successful provers (e.g., mathematicians, strong students) and unsuccessful provers (e.g., typical students) use diagrams and examples is one avenue for addressing this question" (p. 14). The papers in this special issue seek to address the voids in both the guidance available to practitioners and the research base upon which such guidance is founded.

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